VOLUME LIII

The Mathematics Teacher

NUMBER THREE

MARCH 1960

The official journal of
THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The Mathematics Teacher *is the official journal of The National Council of Teachers of Mathematics devoted to the interests of mathematics teachers in the Junior High Schools, Senior High Schools, Junior Colleges, and Teacher Education Colleges.*

All editorial correspondence, including books for review, should be addressed to the Editor.

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THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N. W., Washington 6, D. C.

Printed at Menasha, Wisconsin, U.S.A. Entered as second-class matter at the post office at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 412, P. L. & R., authorized March 1, 1930. Printed in U.S.A.

The Mathematics Teacher

volume LIII, *number* 3 March 1960

THE MATHEMATICS TEACHER is published monthly eight times a year, October through May. The individual subscription price of \$5.00 (\$1.50 to students) includes membership in the Council. For an additional \$3.00 (\$1.00 to students) the member may also receive *The Arithmetic Teacher.* Institutional subscription: \$7.00 per year. Single copies: 85 cents each. Remittance should be made payable to *The National Council of Teachers of Mathematics,* 1201 Sixteenth Street, N. W., Washington 6, D. C. Add 25 cents for mailing to Canada, 50 cents for mailing to foreign countries.

A student experience-discovery approach to the teaching of calculus

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Students of calculus experienced the thrill of discovery and possessed superior knowledge of the theory of calculus when taught by the methods discussed in this article.

Introduction

In the last several decades there has arisen much concern over the importance of understanding and meaning in mathematics as developed by the student himself. One need only refer to such studies as those of Fawcett,[1] Sobel,[2] Schaaf,[3] and others to note experimentation at the high school level on encouraging a maximum of student discovery and student organization of materials. Results of these investigations suggest that the student gains deeper understanding when he is permitted to organize mathematical ideas in his own way and that this is accomplished with no loss of problem-solving proficiency. Too, and perhaps most important of all, the student who seeks to defend *what he himself has formulated* experiences the deep-seated joy of creativity and develops a realistic insight into how mathematics grows.

Origin of the study

In attempting to encourage advanced mathematical study by interested and able students the writer devised a one-semester course in polynomial calculus as a part of a four-year program in secondary mathematics.[4] Succeeding years brought forth improved study guides until it became possible, it seemed, for the students to see calculus in its unveiled beauty and clarity and with none of the mystery that so often shrouds the heart of the subject and causes manipulations to be performed in a meaningless way.

Indeed, it has been found that this captivating problem of *teaching* the calculus has been of deep-felt interest for some time. As early as 1907 Saxelby, in *A Course in Practical Mathematics*, recognized the problem of helping students *understand* the calculus, and in writing of his approach to differentiation suggested that "it too often happens that a student . . . acquires a merely fatal facility in differentiation, regarding it as a mechanical juggling with symbols but having no conception of its relation to experience."[5] Saxelby further urges the teacher to consider the immense value of experimental or numerical or arithmetical or graphical methods and then adds that

> . . . this intuitional direct vision method is intended, not to take the place of, but to prepare the way for, a more rigorous analytical study of the subject. . . . The most natural method of advance is by a series of successive approximations to logical rigor, and, in fact, this is the way in which the subject has actually grown up.

[1] Harold P. Fawcett, *The Nature of Proof*, Thirteenth Yearbook (Washington, D. C.: National Council of Teachers of Mathematics, 1938).

[2] Max A. Sobel, "Concept Learning in Mathematics," The Mathematics Teacher, XLIX (October 1956), 425–30.

[3] Oscar Schaaf, "Student Discovery of Algebraic Principles as a Means of Developing Ability to Generalize" (Doctoral dissertation, Ohio State University, 1954).

[4] New Washington (Ohio) High School.

[5] F. M. Saxelby, *A Course in Practical Mathematics* (London: Longmans, Green, and Company, 1907), p. v.

> ... The process by which the science itself was formed is also the most natural for the mind of the student.[6]

Pólya urges that we examine a second, often ignored, facet of mathematics—"mathematics *in statu nascendi*."[7] He says that "mathematics in the process of being invented has never before been presented in quite this manner to the students, or to the teacher himself, or to the general public."[8]

Teacher-scholars from both the past and the present have sounded a call for a challenging experiment. Is it really possible to conduct a planned course in beginning calculus at the faster-paced university level and to utilize as much as possible discoveries of students themselves? Will such an approach increase understanding? The remainder of this article is largely a story of an attempt to test the hypothesis that a student experience-discovery approach to calculus in the university yields results every bit as promising as those under similar conditions on the secondary level.

The study planned

Accordingly an experiment was planned which would put this hypothesis to test. First, it was necessary to arrange a sequence of topics for a beginning-quarter course in such a way that the contributions of students could be advantageously utilized. Materials were arranged to develop understanding in the use of some of the fundamental ideas before these concepts were subjected to critical discussion by the class or before results suggested as hypotheses were finally deduced. One section of the course was devoted to developing the calculus as a deductive system.

Secondly, there was constructed a series of study-guide sheets containing questions, suggestions, and exercises by which the student, either independently or with the help of class discussion, could arrive largely by his own efforts at some methods and facts of the calculus.[9] It was planned that the text for the course, Hollis R. Cooley's *First Course in Calculus*,[10] would be used for problems other than those of the study guides, as a mathematics source book, and/or as a means for the student to compare "his own book" with that of another author.

It was also desirable to arrange for experimental and control sections in the first-quarter calculus. The one-quarter experimental sections were conducted for two quarters in an atmosphere rich in encouraging discovery, whereas the control groups were taught more or less traditionally by capable men of long university teaching experience. The same text was used in all sections.

To provide some means for comparison of the two experimental and the two traditional sections and to prepare for a statistical study of results, the investigator used a calculus pretest constructed by the writer, the American Council of Education (ACE) Psychological Test, and the average grade in previous mathematics courses.[11] For brevity, tables of comparison are not included in this article.

At the close of the quarter each section took the two-hour examination made by the experimenter for his section and the examination made by the other section's instructor for his section. Further evaluation of the experimental course was attempted by the use of written student reactions and by noting student thought in a short paper on "What I Conceive the Calculus to Be."

At the beginning of the course, there was a short discussion of situations and problems which involve rates of change. Quite immediately the class was con-

[6] Saxelby, *op. cit.* pp. v, vi.

[7] G. Pólya, *How to Solve It* (Princeton: Princeton University Press, 1946), p. vii.

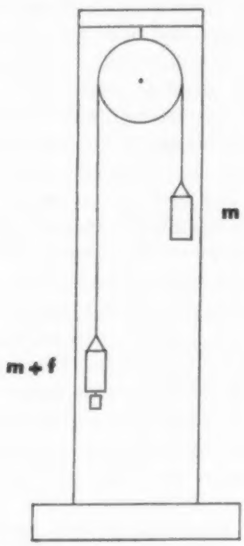
[8] *Ibid.*

[9] During the writer's talk at the 18th Christmas Meeting of the National Council of Teachers of Mathematics, sample copies of some of these study guides were distributed. Other requests were filled by mail.

[10] Hollis R. Cooley, *First Course in Calculus* (New York: John Wiley and Sons, Inc., 1951).

[11] A few other comparisons were made involving, among other things, the time gap between the study of analytical geometry and calculus, age, military experience, and attitude toward mathematics.

fronted with computing the speed of the body labeled $(m+f)$ in the apparatus shown in the figure as it fell under the action of a force f. The speed at the end of the second second $(t=2)$ was chosen for consideration. Data were taken as members of the class volunteered to mark distances fallen at the end of each second as determined by a swinging pendulum. The group suggested that quotients of smaller and smaller distances from the point at which $t=2$, divided by corresponding time intervals such as $\frac{1}{2}$ sec. or $\frac{1}{4}$ sec., would give better approximations to the speed at $t=2$. The difficulty of marking distances at times indicated by pendulums swinging with increased rapidity, however, soon provided convincing evidence that this method was not very feasible.



This difficulty led to the acceptance of *idealized expressions* to describe nonuniform motion such as the well-known relation $s=16t^2$. The students suggested the same method of attack, only this time average speeds were computed by the use of a *formula* to arrive at a sequence of approximations to the speed at $t=2$ sec. In fact, using times $t=2$, $t=2\frac{1}{2}$, $t=2\frac{1}{4}$, $t=2\frac{1}{8}$ secs. $\cdots$ to compute distances through which the body fell at the end of these times, and from these distances to arrive at average speeds during various time intervals, one obtains a sequence of average speeds 72, 68, 66, 65, $64\frac{1}{2}$, $\cdots$ feet per second. The answer to the problem suggested by this sequence is 64 feet per second. Here it was emphasized that we did not know (*a*) if this sequence of average speeds really converged, (*b*) what the "limit" really was, or (*c*) if this "limit" was the answer to our problem. This uncertainty nevertheless gave rise to some hypotheses and definitions, and a logical structure was begun, although it was felt that changes might come upon re-examination. Numerical exercises were suggested that investigated in an introductory way other manners of making the time interval smaller.[12] Some quite ingenious methods were suggested and worked out by the students. Student imagination, student discovery, and student contributions frequently reached a pleasing peak.

To lessen the amount of numerical computation which had served well to develop fundamental ideas, the average speed from $t=2$ to $t=2+h$ was used. Then the speed at $t=2$ was taken to be the

$$\lim_{h\to 0}\frac{16(2+h)^2-16(2)^2}{h},$$

and the students gained practice in a first step in abstraction.

From student suggestions in the various groups actually came the idea that one might compute the speed at *any* time $t=t_1$ and hence work "many problems in one" by this procedure. Speeds for various times and for various expressions of distance could then be found by replacing t_1 by some numerical value for time.

Of course this speed problem was carefully examined, and at the same time the problem of computing the slope of a curve

[12] In one case, $t=2$ was approached through the times $t=3$, $t=2\frac{1}{2}$, $t=2\frac{1}{4}$, $t=2\frac{1}{8}$, $\cdots$; in another case, the sequence was $t=3$, $t=2.1$, $t=2.01$, $\cdots$: in still another, $t=1$, $t=1\frac{1}{2}$, $t=1\frac{3}{4}$, $t=1\frac{7}{8}$, $\cdots$.

was considered. It did not take long for the students to recognize elements common to the two problems, and from this consideration emerged the general "rate-of-change" idea. The result of computing this rate of change of y with respect to x was indicated by R_xy.

The discovery of ways to compute R_xy at sight, various uses of R_xy, the properties of R_xy at maximum and minimum points on a curve, and the use of R_xy to approximate changes in y relative to a given change in x at $x=x_1$ were all developed by inductive exercises and discussion on the guide sheets.

Not until the more critical study of R_xy and its definition as the limit of a sequence of quotients,

$$\frac{f(x_1+h)-f(x_1)}{h},$$

as $h \to 0$, was the symbol changed to that of D_xy.

In like manner, the idea of the *integral* had a humble numerical beginning. An intuitive notion of area under a curve was accepted, and attempts were made to find the area bounded by a curve such as one whose equation is $y=x^2$, $x=1$, $x=3$, and the x-axis. Some students suggested the use of sums of areas of rectangles to make approximations, but it seemed that students who attempted to use the sums of areas of *trapezoids* fared better in ease of setting up numerical sequences to lead to a number which they would present as the area. Results of the use of trapezoids are shown in Table 1. As a class exercise, it was noted that the sum of areas for any number of trapezoids could be expressed as

$$S=10-(1+1/4+1/16+1/64 \cdots)$$

and that this could be evaluated by the aid of the well-known formula for the sum of an infinite geometric progression, $S_\infty = a/(1-r)$, whence $S=8\frac{2}{3}$ square units. Although the use of trapezoids was preferred in computational work, area was *defined* as the limit of a sequence of sums of areas of rectangles as the number of rectangles increased and as the widths of these rectangles decreased. In a similar way were considered problems of "work done" when the force is variable.

In the critical study of these concepts students noted that area and work as defined are both limits of sequences of sums of *products* (analogous to *quotients*, which lead to the derivative), and from these two ideas a definition was abstracted for a concept for which, as yet, we had no name! For some time this was called the "area-computing" process, and a symbol such as $A(x^2)\big|_0^2$ was used to signify area (and also work).

As the course progressed, the usual theorems on sequences, limits of functions, and derivatives were deduced. The limit of a sequence of sums of products was given the name "integral," and theorems on integrals were formulated. From the very definition of the integral, the values of

$$\int_a^b k\,dx, \quad \int_a^b kx\,dx, \quad \text{and} \quad \int_a^b kx^2\,dx$$

were worked out and listed as theorems. It was then *hypothesized* that

$$\int_a^b kx^n\,dx$$

had the value

$$\frac{k(b^{n+1}-a^{n+1})}{n+1}.$$

Striking relations between these results and the antiderivative of kx^n were discovered and noted by the students

TABLE 1

SUMS OF AREAS OF TRAPEZOIDS IN APPROXIMATING (INTUITIVE) AREA UNDER $y=x^2$ FROM $x=1$ TO $x=3$

Number of trapezoids used	*Sum of areas of these trapezoids*
1	10
2	9
4	8 3/4
8	8 11/16
16	8 43/64

throughout all this study, and a feeling developed that some great fundamental link between the derivative and the integral was in the making. This sense of "something about to happen" came to a climax with the fundamental theorem of the calculus

$$D_t \int_a^t f(x)dx = f(t),$$

for $f(x)$ continuous in $a \leq x \leq t$, and with the corollary that

$$\int_a^b f(x) = F(b) - F(a)$$

where $D_x F(x) = f(x)$.

It should be mentioned that topics were so timed in the planning of the work that while some ideas were being refined through discussion and investigation other new thoughts were being introduced at a more elementary level; hence several different topics were under consideration at the same time. This was done to encourage careful and slow nurturing of ideas.

Student growth and contributions throughout the unfolding of the calculus created an inspiring morale in the group as the classroom became a center for discussion, practice, drill, guidance, and investigation. Several of the definitions, hypotheses, and theorems were designated by the names of students who first suggested or deduced them, to help to emphasize how mathematics grows.

Resulting content

Although a first-quarter course in the calculus had been previously outlined, some minor changes resulted while the work was conducted under this student experience-discovery approach. As one would expect, there were really two facets in content: (*a*) the sequence of topics studied, and (*b*) some realizations of more intangible nature—an awareness of logical structure in the calculus and the role of inductive and deductive approaches to mathematics. The resulting topical content is shown in Table 2.

TABLE 2

Table of contents of experimental course

I. *Problems of the Calculus*
  A. The General Rate Problem
    1. Estimating experimentally the speed of a falling body. Exercises.
    2. Estimating the speed when a formula for distance is known. Exercises. Fundamental terms. Exercises.
    3. Estimating speed with less numeriical computation. Functions. Limits.
    4. Estimating speed at any time t_0. Exercises.
    5. Finding the slope of a curve. Graphs. Maxima and minima. Exercises.
    6. General rate of change. Functions. General picture. Exercises.
    7. Uses of changing rates. Related rates. Errors. Small changes.
    8. General review exercises.

  B. The General Area Problem
    1. Approximating area (intuitive) under a curve. Numerical methods. Other methods. Exercises.
    2. More work on areas. Exercises.
    3. Approximation of work done on a spring. Exercises.
    4. Use of approximating *functions*. Exercises.

  C. The Fundamental Problems Recapitulated

II. *A Systematic Look at the Calculus*
  A. Sequences. Limits. Theorems.
  B. Limits of functions.
  C. Continuous functions.
  D. Review.
  E. General review exercises.

III. *The General Rate Problem Treated*
  A. The Derivative
    1. Definition developed from a previous process. Some theorems. Exercises.
    2. Making differentiation easier. Differentiation of products, quotients.
    3. Differentiation of a function of a function. Exercises.
    4. Inverse functions. Parametric functions.
    5. Implicit functions.
    6. Operation inverse to differentiation.
    7. General review exercises.

  B. Second and Higher Derivatives
    1. Derivatives of derivatives and their interpretation. Use.
    2. Derivatives and the study of rectilinear motion. Exercises.
    3. Review exercises.
    4. Higher derivatives of parametric functions.
    5. Higher derivatives of implicit functions.

C. Inverse Rate Problem Continued
   1. How can inverse differentiation be used?
   2. Given slope to find equation of a curve. Finding the length of a curved segment.
   3. A general formula for antidifferentiation of ax^n.

IV. *The General Area Problem Treated*
   A. The Definite Integral
      1. Defined as a name for something we have developed before. Theorems. Refining our thinking. Fundamental terms.
      2. Fundamentals, theorems, applications.

   B. Evaluation of Integrals
      The mean-value theorem. Applications of integrals. The fundamental theorem of the calculus.

   C. Some Applications of Integrals
      Areas. Work. Volumes. Force of liquids on a vertical wall.

V. *General Review*

Throughout the course there was much emphasis on delineating undefined terms, defined terms, hypotheses or postulates, and deduced statements or theorems. In fact, in every experimental section some fundamental philosophical questions often arose, such as: "Is calculus based upon things 'approaching'—if so, isn't this a rather weak foundation?" Here was an opportune time to discuss again the structure of mathematics. One student commented aptly that "Our faith that a certain number is the answer to a speed or to an area problem is just as firm as our faith that a certain sequence approaches a certain limit." On another occasion there arose the question of distributivity of the "area-computing process"—is $A(x^2+x^3) = A(x^2)+A(x^3)$? The study of this question was another rewarding experience in the group's mathematical adventure and development.

Each student kept a notebook that he organized in his own way. A central core of hypotheses, definitions, and theorems, however, was quite common to all such efforts.

The *skills* developed in traditional first-quarter calculus were nurtured in this course also. In this approach, however, the student gained more practice in *numerical work* as he was being guided to abstractions.

Evaluation

As indicated before, several kinds of evaluation were attempted. The cross-test made for the experimental sections and taken by both the experimental and traditional groups consisted of short-answer items and discussion questions that were designed to measure understanding. It also contained problems. The test made for the traditional sections by their teachers, and taken by both groups, largely contained problems. Each teacher used these tests in whatever manner he wished for his own course. The tests were mixed into a packet and graded uniformly by the writer. The results of this grading are the scores which appear in this study. Results from the winter quarter cross-testing program are found in Table 3.

TABLE 3

ARITHMETIC MEANS OF SCORES ON TESTS (WINTER QUARTER)

Section taking test	*Score on Test One made for Experimental Section One. Total score possible 272*	*Score on Test A made for Traditional Section A. Total score possible 180*
Section One (experimental)	217	130
Section A (traditional)	131	130

It seemed advisable to give further study to these final test results in the light of initial data on previous grades, the pretest, and the ACE scores, in an attempt to make some conclusions on performance independent of the initial data. Accordingly, data of the traditional and experimental groups were combined to develop regression equations which would predict for each student an expected score on Test One and on Test A on the basis of the

three preliminary test scores of both sections. In fact, it turned out that the ACE and calculus pretest scores, at least as used here, had no significant effect on the Test One and Test A scores. The results showed that:

a. The students in the experimental group scored on the average 27.10 points higher on Test One than would be expected on the basis of their preliminary test scores (significant at the 1% level).
b. The students in the traditional group scored 51.59 points lower on Test One than would be expected (significant at the 1% level).
c. The differences were not significant on Test A.

The results indicate that the method of teaching under examination was especially effective in promoting a deeper understanding of the calculus and that this gain was not at the sacrifice of proficiency in manipulations and applications. It is possible that these differences in achievement were due to factors unknown to the experimenter, but the statistical analysis indicates that they were not due to student variations in previous grades, mathematical preparation as measured by the calculus pretest, or innate ability as indicated by the ACE scores.

Since the results from the winter quarter were so positive, a statistical study was not made on the test scores at the close of the spring quarter. The data "in the raw" indicated similar results, however.

In addition to statistical studies on cross-tests of proficiency and understanding, there was an attempt to evaluate the effectiveness of this approach to the calculus by searching out student reaction on certain items. Some of these items are listed in Table 4.

Inquiries were also made on student awareness of logical relation in the calculus, on experiencing how mathematics grows, on sensing the power of the calculus method, on the enjoyment of mathematics, and on whether, and at what points.

TABLE 4

NUMBER OF SATISFACTORY RESPONSES TO SOME TEST ITEMS ON LOGICAL RELATIONS

Test item	*Nature of expected response*	*Number of satisfactory responses from*	
		Experimental Sections	*Traditional Sections*
$D_x y = \lim\limits_{h \to 0} \dfrac{f(x_1+h)-f(x_1)}{h}$			
What is the limitation placed upon h?	$h \neq 0$	36	10
The relation $D_x(f+g)$ comes from what basic theorem on limits?	$\lim\limits_{n\to\infty} (a_n+b_n)=A+B$	24	7
What would you suggest as the concept most basic to the calculus?	Sequences or limits	24	5
Deduce the quotient rule, the product rule, or the chain rule.	(A correct deduction)	25	1
Explain briefly or describe the logical thread which connects the process of differentiation with that of integration.	Antidifferentiation and the fundamental theorem	18	0
Total number of responses.		38	24

the student experienced a feeling and thrill of discovery. Without attempting to classify them in this article, sundry quotations taken from student responses will help communicate student reaction:

1. "They (the derivative and the integral) both have the same beginning in the theories of limits. They are both limits, only the derivative is the limit of a sequence of quotients and the integral is the limit of a sequence of sums of products."
2. "We start with sequences and go from there to the [idea of] rate of change, and then to the derivative and the antiderivative. On the other side of the picture we go from sequences up through areas to the integral. Here we go through the mean-value theorem for integrals to the fundamental theorem. The fundamental theorem, therefore, is the link which holds the two chains together."
3. "I could see the calculus grow from the beginning arithmetic methods to the more involved theorems."
4. "Our ideas started on a 'man-on-the-street level' and then grew."
5. "The completeness of our calculus structure impresses me. It is to me like building a house—afterwards I don't just look at the roof, but at all the beautiful parts of the house under the roof."

Perhaps the most important measure of all is the student evaluation of this student experience-discovery approach. A summary of responses is given in Table 5.

TABLE 5

A SUMMARY OF FIFTY-THREE RESPONSES TO "DO YOU THINK THIS APPROACH IS REALLY DIFFERENT?"

Student Responses	*Frequency*
I understand *reasons* for doing, rather than doing these mechanically	22
New work is developed from the student from his own knowledge	9
Students helped to develop the course; were not merely passive in acceptance	5
Class more "human"; better student-teacher relation	4
We saw how things develop and we co-operated to bring out a new idea	4
We saw a problem and then saw progressively simpler methods to find the answer	4
This approach gives one a good beginning	3
My interest has been aroused more than ever before	2

Some statements seem to convey more than it is possible to suggest when they are condensed in a summary table:

1. "[This approach is different] because we really got to the very basic beginning which is something I have really not done before; most other courses are such that we work a lot of problems with very little theory."
2. "It stresses the reasons and theory behind the problems. This is something *I* have never seen and have always wanted in a mathematics course."
3. "[This approach is different] mainly because it is the first time in all my mathematics that the very basic material was taught so for once I know what I was really studying."
4. "It is different in many ways. The most significant is the fact that the student is helped to discover the principles for himself, thus giving a clearer understanding."
5. ". . . this makes [the calculus] a part of him [the student] and it stays alive for him . . . this is the best understood course, the finest, and the most clear course of any kind that I have ever taken."
6. "It used to be that I did not know how they got what they got in mathematics, but now I do."
7. "This approach makes the mathematics 'alive.' It is living and growing! Some students I talk to just differentiate—they do not know what they are doing, nor do they understand what a derivative is."
8. "This method seems to take the magic or mystery out of mathematics."

SUMMARY

It is generally agreed that concern for understanding rather than manipulative skill alone is a matter of utmost importance in encouraging the attainment of the quality which a mathematics education should provide.

Hence there is proposed in this study a student experience-discovery method whose methodology has characteristics as follows:

1. General problems find their initial setting in natural situations.
2. The students attack these problems with the means they know—often arithmetic.
3. Results are studied, examined, discussed; definitions and hypotheses are made.
4. Abstract considerations flow naturally out of the concrete.
5. The teacher guides the student to the consideration of more profound questions.
6. Statements accepted as hypotheses remain so until deduced as theorems with what is being *assumed* clearly delineated and set apart from what is being deduced.

7. Student initiative, discovery, and suggestions are encouraged and utilized to help develop the calculus.

A test of this method in the classroom results in some tentative "good news" for the student. On the basis of this experiment there can be proposed the working hypothesis that students taught by this "discovery" approach have these advantages:

1. They do as well on problems and manipulative skills as those with traditional instruction but, in addition, they have increased understanding.
2. They possess a superior knowledge of the fundamental theory and logical relations among parts of the calculus.
3. They experience the thrill of discovery and the satisfaction of producing results through creative effort—all of which lead to greater enjoyment of mathematics and a deeper understanding of its nature and use.
4. They express ideas of the calculus in their own language and they undergo the stimulating and disciplinary experience of having their expressions and ideas sharpened through examination by other students as well as by the teacher.

Since one can hardly present the results on *one* experiment as established conclusions, the writer suggests that the advantages of the student experience-discovery approach be investigated further. To continue the teaching of the calculus by methods which are antithetical to those of the experience-discovery approach *may* be depriving our students of a richness of mathematical education which they deserve and which it is so imperative that they have.

Have you read?

FORSYTHE, GEORGE E. "The Role of Numerical Analysis in An Undergraduate Program," *The American Mathematical Monthly*, October 1959, pp. 651–62.

"Where do mathematics students go?" This is the first sentence of this article. Although written about the college program, it sets some directions for the elementary and high school mathematics programs. The A.B. graduate in mathematics is lured to the computers; therefore, his program of preparation must include a knowledge of the structure of mathematics, an increasing ability to study mathematics independently, a knowledge of where to find things he does not know, an ability to solve original problems, a knowledge of related fields, and an increasing joy in the study of mathematics.

Besides this, he must be taught with a balance between the abstract and the concrete; he must be aware that the whole subject flows from a few postulates and principles.

Numerical analysis must be a part of his preparation. This might well be called today the art and science of using digital computers. He must learn to code, to know limitations of computers, and to be aware of errors. There are many examples of where these items may be taught. Read this article and ask yourself, "Are my high school graduates ready?" You may want to read "A New Kind of Course for the Preparation of Teachers" in this same issue.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

PARGETER, A. R. "Plaited Polyhedra," *The Mathematical Gazette*, May 1959, pp. 88–101.

Many of you have used models of polyhedra for teaching purposes and for motivation. This article will interest not only you but your students since it is not the common approach to polyhedra construction. "Plaited" refers to the kind of things girls may do to their hair or boy scouts do with leather thongs. It would be unwise to attempt to show how this is done, but this article explains in some detail the process to use and the best material from which to make plaited polyhedras. The icosohedron is considered the easiest to make and should probably be the first trial. The author provides the theory. For example, when two strips cross, the region has four vertices and therefore is a quadrilateral. You should read the article, get some paper, and try it. You will have fun and so will your students. In this same issue of the *Gazette* is an interesting article of one page on $\times$, the sign for multiplication, which you may like.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Linear programming problems for first-year algebra

DONOVAN LICHTENBERG *and* MARILYN ZWENG,
University of Wisconsin, Madison, Wisconsin.
The description of a unit on linear programming taught at Wisconsin High School, Madison, Wisconsin, 1958–59.

NEW APPROACHES in mathematics make possible the teaching of certain topics not previously included in high school mathematics. Linear programming problems form an example of this trend.

Certain maxima and minima problems which arise in business, industry, and in the military are called linear programming problems. We believe that students in first-year algebra are not only capable of handling the simpler types of such problems, but that they also find these more interesting and worth-while than the verbal problems usually taught in traditional algebra courses. The reader of this article will realize that the insertion of such material in the first-year algebra course will require prior development of some concepts not generally taught in the traditional course.[1] The following is an example of a linear programming problem:

> A manufacturer makes two models, A and B, of a product. Each model must be processed by two machines. To complete one unit of model A, machine I must work 1 hour and machine II must work $2\frac{1}{2}$ hours. To complete one unit of model B, machines I and II must work 4 hours and 2 hours respectively. Machine I may not operate more than 8 hours per day, and machine II, not more than 12 hours per day. If the profit on model A is \$3 per unit and on B, \$4 per unit, how many units of each model should the manufacturer produce per day to maximize his profit?

Before we discuss the solution of this problem, let's investigate some of the theory involved. We are going to be dealing with functions defined by equations of the form $f(x, y)=ax+by+c$. The domains of our functions will be sets of ordered pairs which can be defined by inequalities. Suppose we have

$$A=\{(x, y)\,|\,y\geq x,\ 2x+y\leq 9,\text{ and }x\geq 1\}.$$

The graph of this set is shown in Figure 1. The boundary lines are included in the set. That is, the set is closed. Mathematicians would call A a polygonal convex set.

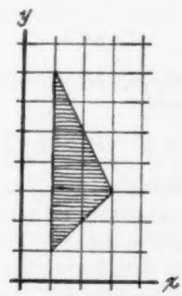


Figure 1

Now think of this set of ordered pairs as the domain of a function defined by $f(x, y)=ax+by+c$. We want to find those ordered pairs in the domain for which $f(x, y)$ attains a maximum or minimum value. We know from analytic geometry that the graph of such a function will be a plane. If $a=b=0$, the graph will be a plane parallel to the xy-plane and c units from it. That is to say that $f(x, y)$ would be equal to c throughout any domain and thus we would not talk about the maximum or minimum of $f(x, y)$ for A. This type of

[1] Robert E. K. Rourke, "Some Implications of Twentieth Century Mathematics," THE MATHEMATICS TEACHER, LI (February 1958), 74–86.

function, then, is of no interest here. Suppose instead that a and b are not both zero. Then the graph of our function will be a plane which is not parallel to the xy-plane. If this is the case, for what member of A will $f(x, y)$ attain its maximum (minimum)? It should be intuitively clear that, since the graph is a plane, the maximum (or minimum) value of $f(x, y)$ will occur either at one of the vertices of the triangle or at all of the points along one side. See Figure 2.

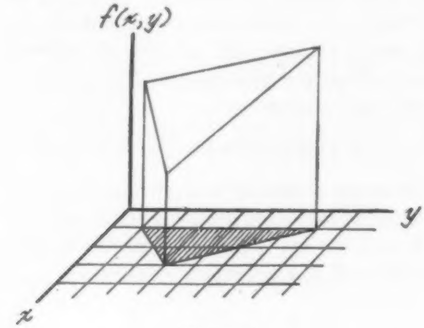


Figure 2

To illustrate this, let us study a specific function. Consider

$$f(x, y) = 2x + 5y - 3$$

where the domain is still A. For what (x, y) in A will $f(x, y)$ attain its maximum? Inspection will show that $f(1, 7) = 34$ and no other (x, y) in A will give $f(x, y)$ a larger value. We also see that $f(1, 1) = 4$, and this is the minimum value of $f(x, y)$. Both the maximum and minimum values of $f(x, y)$ are attained at corner points of A.

The preceding discussion is summarized in what we will call the fundamental theorem of linear programming.

If we have a linear function f *whose domain of definition is a convex polygonal set, the maximum (or minimum) value of* f(x, y) *is attained at one of the corner points of the convex set or at all of the points along one side.*[2]

[2] For proof of this theorem, see John G. Kemeny, J. Laurie Snell, and Gerald L. Thompson, *Introduction to Finite Mathematics* (Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1957), pp. 256–57.

Let us now return to the problem about the two models of a product. Let x represent the number of units of model A to be produced and y the number of units of model B. Certainly, x and y cannot be negative numbers. Therefore, two conditions which must be satisfied are $x \geq 0$ and $y \geq 0$. Now, model A requires 1 hour on machine I and model B requires 4 hours on this machine. Since machine I cannot be run for more than 8 hours, we have the condition

$$x + 4y \leq 8.$$

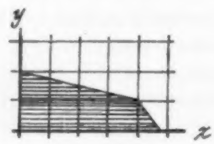


Figure 3

The models A and B require, respectively, $2\frac{1}{2}$ and 2 hours on machine II, which cannot be run for more than 12 hours. Thus, we have the condition

$$\tfrac{5}{2}x + 2y \leq 12.$$

Now, consider the set M of all ordered pairs (x, y) which satisfy all four conditions:

$$M = \{(x, y) \mid x \geq 0,\ y \geq 0,\ x + 4y \leq 8,\ \tfrac{5}{2}x + 2y \leq 12\}.$$

The graph of M is shown in Figure 3.

Set M has all the possible solutions to the problem. That is, none of the ordered pairs outside M satisfies all of the given conditions. The problem now is to determine which (x, y) in M maximizes the profit. We see that the total profit in dollars for a day will be $3x + 4y$. The fundamental theorem tells us that we only have to test the four corner points.

At $(0, 0)$, $3x + 4y = 0$

At $(0, 2)$, $3x + 4y = 8$

At $\left(\frac{24}{5}, 0\right)$, $3x + 4y = 14.40$

At $(4, 1)$, $3x + 4y = 16$

Testing these four points, we find that production of 4 units of model A and 1 unit of model B gives a maximum profit of $16 per day.

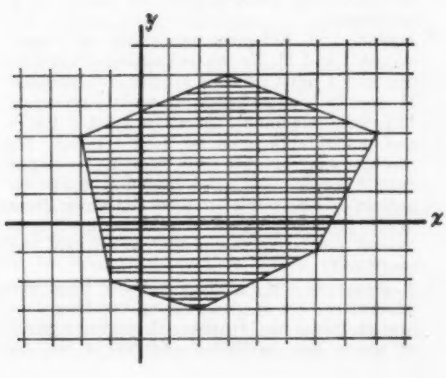
Now how was all of this handled in first-year algebra? The theory of linear programming was taught to ninth graders, not as a rigorous theory, but rather from an intuitive approach. It is obvious that an ability to handle inequalities was a necessary prerequisite for this unit. From the beginning of the year our students at Wisconsin High had worked simultaneously with inequalities and equations, so they were well prepared for the graphing which was to follow.

The purpose of our first assignment was to lead the students to the discovery of the fundamental theorem on convex sets; that is, that the maximum and minimum of any function whose domain is defined by a convex set is at the corner points of the set. The first assignment was similar to this:

> The shaded area in Figure 4 is the graph of a set which we shall call S. Tell which ordered pairs in S will give the following expressions their maximum and minimum values.
> (*a*) $3x+4y$
> (*b*) $3x-4y+1$
> (*c*) $x+3y$
> (*d*) $2x-7y+4$
> (*e*) $9y-3x$
> (*f*) $-6x-y$

The students were simply to evaluate the given linear expressions for various points in the set. Mathematical patterns had been emphasized throughout the year, so the students rather suspected that some pattern would occur here. It was not long before they began to notice the relationship between the maxima and minima, and the corner points (or sides) of the given set.

Figure 4



We made no further attempts to prove the fundamental theorem rigorously. We established our principle merely by providing many experiences similar to the example above.

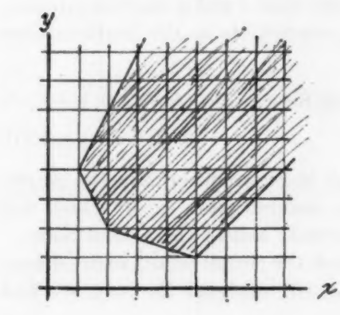
You will notice that we did not use the term "function" in this assignment, nor did we use it later in the unit. The word "expression" sufficed for our purposes, and yet you can see that this material provided an informal experience with the function concept.

We had further assignments which illustrated other important facts. If the convex set is not bounded, the linear expression may attain a maximum or a minimum, but not both. Consider, for example, the set T where

$$T=\{(x,y)\,|y-2\leq 2x, 2x+y\geq 6, 3y+x\geq 8, \text{ and } x-y\leq 4\}.$$

The graph of this set is illustrated in Figure 5. The point (1, 4) will give the expression $2y-5x$ a maximum value of 3, but there is no minimum. The point (5, 1) will give the expression $x+4y$ a minimum value of 9, but this time there is no maximum. A couple of the better students in

Figure 5



class observed that in this situation, where our set is unbounded, we could have an expression with neither a maximum nor a minimum. (Consider the expression $3x-2y$.)

The following is an illustration of the most difficult type of problem with which we worked:

An appliance dealer has stores in Milwaukee, Madison, Beloit, Beaver Dam, and Fort Atkinson. He has 8 extra refrigerators in Milwaukee and 6 extra in Madison. He would like to move 5 of them to Beloit, 5 to Beaver Dam, and 4 to Fort Atkinson. The transportation costs per refrigerator between the cities are given in the table:

	Beloit	*Beaver Dam*	*Fort Atkinson*
Milwaukee	\$16	\$10	\$15
Madison	\$10	\$12	\$10

How should the refrigerators be distributed to keep transportation costs at a minimum?

It appears at first glance that we need to use more than two variables in this problem. However, we can let x represent the number of refrigerators to be moved from Milwaukee to Beloit and then $5-x$ will be the number to be moved from Madison to Beloit.

If Beaver Dam gets y refrigerators from Milwaukee, it will have to get $5-y$ from Madison. The Fort Atkinson store will get the remainder, i.e., $8-x-y$ from Milwaukee and $6-(5-x)-(5-y)$ or $x+y-4$ from Madison. The total cost of moving the refrigerators will be $16x+10y+15(8-x-y)+10(5-x)+12(5-y)+10(x+y-4)$ or $x-7y+190$. Certainly, the number of refrigerators to be moved from one city to another cannot be negative. Except that x and y must be integers, the only constraints in the problem then are

$$x\geq 0, y\geq 0, 8-x-y\geq 0, 5-x\geq 0, 5-y\geq 0, \text{ and } x+y-4\geq 0.$$

The graph in Figure 6 contains the graphs of all the ordered pairs (x, y) which will simultaneously satisfy these conditions.

We check the corner points to determine which one will minimize the cost. We find

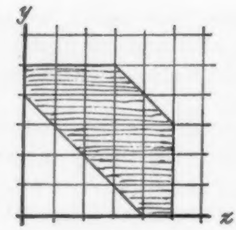


Figure 6

that $(0, 5)$ gives the expression $x-7y+190$ a value of 155, and this is the minimum. Hence if the refrigerators are distributed as in the table below, the appliance dealer will have minimum transportation costs of \$155.

	Beloit	*Beaver Dam*	*Fort Atkinson*
Milwaukee	0	5	3
Madison	5	0	1

It is surprising how well the students guess the solution to this type of problem. After working the problem, they are elated to discover that they have guessed correctly.

Here are some more of the problems our students solved.

1 Two men, Joe and George, are working in a small factory. Since they are both part-time help, they may each be employed for any number of hours. Joe's wages are \$2 per hour. George receives \$3 per hour. Joe produces 10 nuts and 4 bolts per hour. George can produce 5 nuts and 7 bolts per hour. A rush order calls for 50 nuts and 50 bolts. Because of the expense of running the machines, no more than 11 man-hours can be spent on the manufacture of these nuts and bolts. How many hours should Joe work, and how many hours should George work to keep the payroll at a minimum?

2 Aspirin and Buf-pirin are made on 2 machines, I and II. To make 1 pound of aspirin, machine I must be used 1 hour and machine II must be used $2\frac{1}{2}$ hours. To make 1 pound of Buf-pirin, machine I must be used 3 hours and machine II must be used $\frac{3}{4}$ hours. No machine may operate more than 12 hours per day. A profit of \$2 per pound is made on aspirin and \$3 per pound on Buf-pirin. How many pounds of each should be made per day to maximize the profit? What will the profit be?

3 A grocer is buying soap powder from the wholesaler. He is considering stocking two brands, Sudso and Brighto. He makes a profit of 10¢ a box on Sudso and 20¢ a box on

Brighto. Consumer buying statistics show that at least 3 times as much Sudso is sold as Brighto. The grocer has, at most, 900 square inches of shelf space for soap. It takes 20 square inches to store a box of Sudso and 30 square inches to store a box of Brighto. How many boxes of each kind of soap should the grocer stock to maximize his profit? What will the profit be?

4 According to a well-known nursery rhyme, Jack Spratt could eat no fat, his wife could eat no lean. Suppose Jack needs at least 4 pounds of lean meat per week, and his wife needs at least 3 pounds of fat per week. Their diet consists of beef and pork. Each pound of beef is .2 fat and .8 lean. Each pound of pork is .6 fat and .4 lean. The Spratts have a very small refrigerator and therefore cannot buy more than 9 pounds of meat per week. If pork costs \$.75 per pound, and beef costs \$1.00 per pound, find out how many pounds of beef and how many pounds of pork the Spratts should buy per week to minimize the cost.[3]

5 A producer has two mixtures of nuts. Each pound of mixture A contains 2 ounces of almonds, 1 ounce of cashews, and 1 ounce of hazel nuts. Each pound of mixture B contains 1 ounce of almonds, 2 ounces of cashews, and 6 ounces of hazel nuts. The remainder of of both mixtures is peanuts. The producer wants to use mixtures A and B to make a bag which will contain at least 8 ounces of almonds, 10 ounces of cashews, and 18 ounces of hazel nuts. If mixture A costs \$1.00 per pound and mixture B, \$1.50 per pound, how many pounds of each should he use to minimize his cost?

6 A manufacturer produces two different models of steam irons, the Deluxe and the Superior. Each model must be processed by three machines, I, II, and III. To complete one unit of each model, the three machines must work the number of hours indicated in the following table.

	Deluxe	Superior
I	1.0	1.0
II	0.6	1.2
III	1.2	0.4

No machine may operate more than 12 hours per day. The profit on the Deluxe model is \$3 per unit, and the profit on each unit of the Superior model is \$5. How many of each unit should be manufactured per day to maximize the profit? What will the profit be?

Many important mathematical principles are dealt with in the work on linear programming. The students can acquire an insight into certain facts which are generally left to calculus or advanced calculus. For instance, an expression such as x^2+y^2-4x can be included in the introductory assignments. The students will discover that when the expression for $f(x, y)$ is not linear the whole theory breaks down. If we examine the expression x^2+y^2-4x with respect to the set that we called S (Fig. 4), we see that the minimum is attained at (2, 0), which is an interior point of S. The maximum occurs at (8, 3). In our work with this type of expression, one of the brighter students (with a little help) grasped the significance of contour lines. He saw that x^2+y^2-4x has the value -4 at the point (2, 0). It has the value -3 at any point on the circle with center (2, 0) and radius 1. At any point on the circle with center (2, 0) and radius 2, the value of x^2+4y^2-4x is 0. See Figure 7.

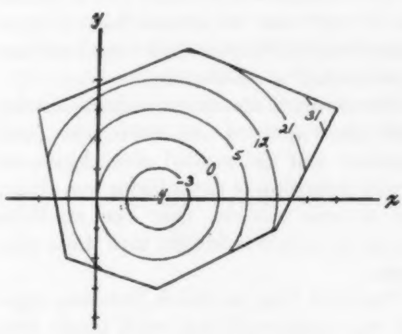
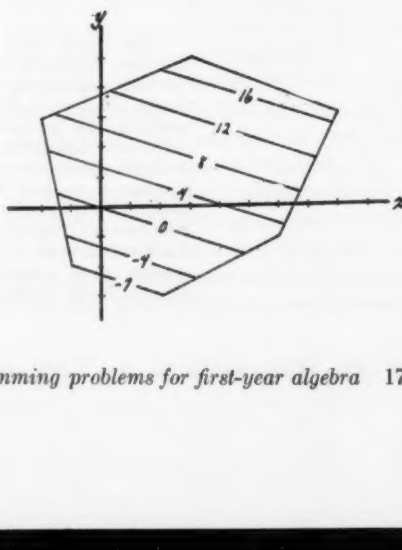


Figure 7

Of course the contour lines for a function defined by a linear expression are straight lines. Figure 8 shows some con-

Figure 8



[3] Problem 4 was suggested by a Jack Spratt problem in Kemeny, Snell, and Thompson, *op. cit.*, p. 263.

tour lines for the expression $x+3y$. Note that any point on the line between $(-1, -2)$ and $(2, -3)$ gives $x+3y$ a value of -7, which is the minimum with respect to set S.

Although we at no time discussed graphing in three dimensions, one boy constructed a device which demonstrated very nicely that the graph of the function defined by $f(x, y)=ax+by+c$ is a plane. He graphed a set of ordered pairs on corrugated cardboard. He then considered the value of a certain linear expression at various points of this set. He erected perpendicular spikes of wire equal in length to the value of the expression at these points. The ends of the spikes then lay in a plane. After this project had been shown to the class, another student suggested that if we had constructed the same sort of device for x^2+4y^2-4x, we would have a cup-shaped object. What he had visualized was a parabaloid of revolution!

The students also became familiar with such ideas as open and closed sets, and bounded and unbounded sets. Again no formal definition of these terms was given, but it was obvious that the students gained an intuitive insight into these concepts.

The work that we did in first-year algebra was concerned only with linear programming problems which involved functions whose domains could be graphed in a plane. That is, the problems could contain only two independent variables. It is important for the teacher to realize that in practical applications problems with three, four, or even twenty variables are met. As we increase the number of variables, the solution becomes more difficult, but the principles remain the same. With three independent variables, the domains of the functions have to be graphed in three-space. The contour lines become contour planes, and the maximum and minimum still occur at corner points (this time of the convex polyhedra). When we have more variables we cannot visualize the problem at all, but applied mathematicians have devised ingenious methods for solution.

For further information on linear programming, the reader is referred to Richardson's *Fundamentals of Mathematics*[4] and Kemeny, Snell, and Thompson's *Introduction to Finite Mathematics*.[5]

Answers to problems

1 Joe should work 2 hours; George, 6 hours.
2 $4\frac{1}{2}$ pounds of aspirin and $2\frac{1}{2}$ pounds of Bufpirin; profit, \$16.50.
3 30 boxes of Sudso and 10 boxes of Brighto; profit, \$5.00.
4 3 pounds of beef and 4 pounds of pork.
5 2 pounds of mixture A and 4 pounds of mixture B.
6 4 Deluxe models and 8 Superior models; profit, \$52.

[4] M. Richardson, *Fundamentals of Mathematics* (New York: The Macmillan Co., 1958), pp. 262–65.
[5] Kemeny, Snell, and Thompson, *loc. cit.*

"On Labor Day, 1959, Dr. B. F. Skinner, Harvard, displayed his teaching machine before the American Psychological Association convention in Cincinnati. Take that as the formal date marking the introduction of what has been called self-instruction or programmed teaching. At least three manufacturers have produced a number of models, not only of Dr. Skinner's device but of several other automatic teaching instruments. The role of automatic teaching in America's schools is the subject of vigorous discussion and research."—*Ben Brodinsky in* Edpress News Letter, *December 18, 1959.*

Linear programming, an aid to decision making

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Some simple examples of linear programming, one of which is in three variables.

LINEAR PROGRAMMING has been described as a way of selecting the best course of action from an infinite number of courses. As one of the rapidly developing aspects of modern applied mathematics, it is concerned with the quantification of decision processes and is used in national defense, industrial production, transportation, personnel work, and a host of other fields. The two-dimensional geometric linear programming model is treated in some elementary texts and popularized writings about present-day mathematics* and is readily visualized by high school students. One rarely sees its three-space analogue, however, primarily because the two-space model adequately illustrates the procedure. Algebraic models for solving higher order linear programming problems are available and, in practice, electronic computing systems carry out the arithmetic. Nevertheless the two-space and the three-space geometrical models are rather simple and afford the high school student who has had a taste of co-ordinate and solid geometry excellent practice in graphing equalities and inequalities in two-space and three-space.

Let's start with a two-space example:

> A furniture manufacturer makes two different types of card tables, the Deluxe and the Regal. Each type must be processed by two machines, Alpha and Beta. To make one Deluxe table, machine Alpha must work 2 hours and machine Beta must work 4 hours. To make one Regal table, machine Alpha must work 4 hours and machine Beta must work 2 hours. The manufacturer can sell all the tables of both types that he desires to produce. His profit on each Deluxe table is \$3; on each Regal table his profit is \$5. How many of each type should he produce daily to realize maximum profit?

Solution: Let x be the number of Deluxe tables and y the number of Regal tables. His profit equation is thus $P=3x+5y$. He wishes to find the values of x and y which maximize P under the following restrictions:

(*a*) $2x+4y\leq 24$

(*b*) $4x+2y\leq 24$

(*c*) $x\geq 0$

(*d*) $y\geq 0$

Restriction (a) asserts that machine Alpha cannot work longer than 24 hours a day, and restriction (b) asserts the same thing about machine Beta. Restrictions (c) and (d) follow from the fact that he cannot produce a negative number of either type. The graph (Fig. 1) of all the points showing restriction (a) is the shaded half-plane below the line $2x+4y=24$ and the graph of restriction (b) is the shaded half-plane below the line $4x+2y=24$. The graph of the intersection of restrictions (a) and (b) is the doubly shaded area.

The addition of restrictions (c) and (d) yields the shaded quadrilateral in Figure 2. Any point in the shaded area, including the boundaries, satisfies all the restrictions. The totality of such points constitutes the

* See M. Richardson, *College Algebra* (Englewood Cliff, N. J.: Prentice Hall, 1958), pp. 387–90, and George A. W. Boehm, *The New World of Math* (New York: Dial Press, 1959), pp. 109–13.

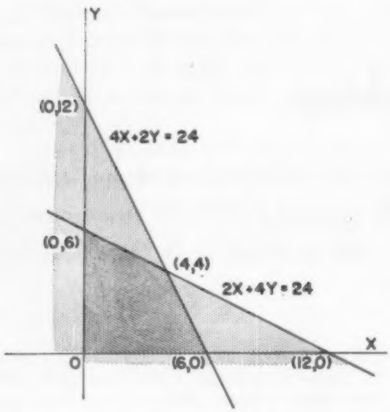


Figure 1

set of *feasible points*, that is, the ordered pairs (x, y) associated with each of the points in this set are all possible solutions to the question of how many of each type of table could be made.

Consider a hypothetical profit, say $P=20$. The profit equation would be $20=3x+5y$, a straight line (Fig. 2). For larger P, say $P=30$, the resulting profit equation has a higher y-intercept (Fig. 2). It can be demonstrated that the profit line with the greatest y-intercept (thus, the greatest profit) and containing a feasible point will intersect the shaded area at a vertex. Therefore, at most, one would check points Q, R, and S to find that if no Deluxe tables and 6 Regal tables are made (point S), $P=30$; if 6 Deluxe tables and no Regal tables are made (point Q), $P=18$; if 4 of each type are made (point R), $P=32$. Thus the manufacturer should make equal numbers of each table to realize the greatest profit.

It is interesting to observe that a hasty decision to produce more Regal tables than Deluxe tables because of the greater profit on Regals would result in a lower total profit.

Now let's try a three-variable linear programming problem:

A drug firm is planning to produce an A and D vitamin compound. They have three basic ingredients, *X*, *Y*, and *Z*, which are available as by-product material from other production The vitamin content and cost per gram of each of these ingredients is tabulated:

UNITS AVAILABLE PER GRAM

	Units of vitamin A	*Units of vitamin D*	*Cost per gram*
Ingredient *X*	120,000	20,000	\$.20
Ingredient *Y*	30,000	10,000	.15
Ingredient *Z*	60,000	200,000	.40

Each bottle of the finished compound is to contain a minimum of 600,000 units of each vitamin. The drug firm wants to determine the number of grams of each basic ingredient to use to meet this specification at a minimum cost.

Solution: Let x, y, and z be the number of grams of X, Y, and Z respectively. The cost equation in cents is thus $C=20x+15y+40z$. It is necessary to find the values of x, y, and z which minimize C under the following restrictions:

(*a*) $x \geq 0$

(*b*) $y \geq 0$

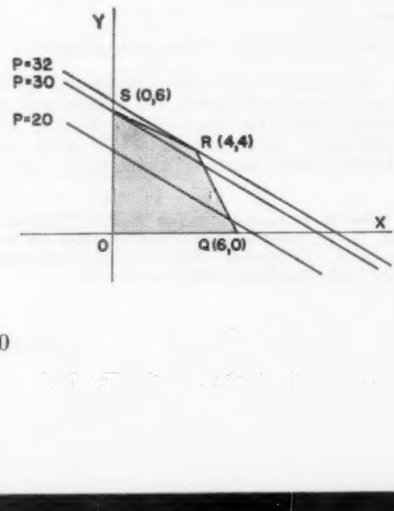
(*c*) $z \geq 0$

(*d*) $120{,}000x+30{,}000y+60{,}000z \geq 600{,}000$

(*e*) $20{,}000x+10{,}000y+200{,}000z \geq 600{,}000$

Restrictions (a), (b), and (c) assert that it is impossible to use a negative amount of any ingredient. Restriction (d) insures that the compound will contain the minimum specified amount of vitamin A; restriction (e) insures the same thing for vitamin D.

Figure 2



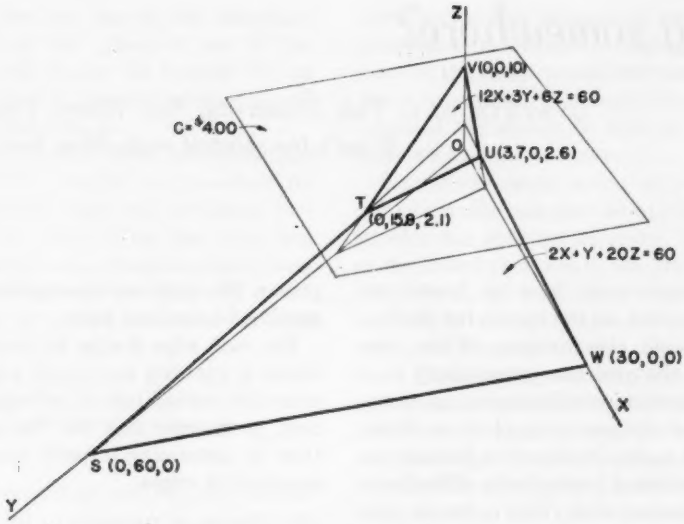


Figure 3

Graphing the restrictions gives us Figure 3.

Restrictions (a), (b), and (c) limit candidacy for the location of feasible points to the first octant. The addition of restrictions (d) and (e) to the space defines the set of feasible points as that portion of the first octant outlined in heavy lines.

In three-space, the possible cost equations comprise an infinite set of parallel planes, one of which is shown in Figure 3 and, analogous to the two-space model, the cost plane which represents the minimum cost contains a vertex of the solid figure of feasible points. Therefore, one would check points S, T, U, V, and W to find the least expensive formulation for the vitamin compound. The co-ordinates of these vertices (correct to tenths) are:

S (0, 60, 0)
T (0, 15.8, 2.1)
U (3.7, 0, 2.6)
V (0, 0, 10)
W (30, 0, 0)

Substituting these values in the cost equation:

Cost at point S	\$9.00
Cost at point T	3.21
Cost at point U	1.78
Cost at point V	4.00
Cost at point W	6.00

Thus the drug firm should use 3.7+ grams of ingredient X and 2.6 grams of ingredient Z, omitting ingredient Y, to minimize the cost of a vitamin compound containing at least 600,000 units each of vitamins A and D.

Clearly, we have looked at very simple examples of linear programming so that the graphing would not be tedious. Real-world linear programming problems are virtually all much more complicated. Each additional restriction increases the number of vertices and each additional variable increases the dimensionality. It is easy to see why strictly algebraic methods are used in lieu of geometric methods as the complexity increases.

Going somewhere?

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What's the shortest route from here to there?

ALMOST EVERYBODY likes to travel on maps, so let this be the excuse for the following simple observations. When one brings out the road map preparatory to a trip, the question usually comes up about the shortest distance to a place or about the fastest route. In regard to distance, it is not considered particularly difficult to find the shortest road. This is due in part to the highway department which provides main roads that connect the main cities, and in part to the fact that we live on a sphere or nearly planar surface. But this latter simplification nowadays tends to be obviated somewhat by modern throughways with limited access where roads may cross on the map without giving any actual opportunity to pass from one to the other.

More difficult is the determination of the route requiring the shortest time. The decision must of necessity depend on considerable experience. Some of the large transport companies make extensive operations in trial runs under varying traffic conditions. For such companies there is, however, another factor which is often decisive, the question of costs. Each run represents expenses in the form of fuel, wages, wear and tear, tolls on bridges and throughways—all to be added up and minimized as far as it may be possible under the circumstances.

Problems concerning minimum distance, time, or expense can all be attacked by the consideration of a graph, or network G, consisting of edges connecting vertices, or branch points. The edges then correspond to the various sections of roads, and the vertices to the forks where they run together. We make no assumption that the graph is located in a plane.

For each edge $E=(a, b)$, there is associated a quantity $e(a, b)>0$ which represents the expenditure in mileage, time, or cost, as the case may be. The question is then to determine a path consisting of consecutive edges

$$(1) \quad P=(a_0, a_1)(a_1, a_2) \cdots (a_{n-1}, a_n)$$

connecting the end points a_0 and $b_0=a_n$ in such a manner that the total expenditure involved

$$(2) \quad e(P)=e(a_0, b_0)=e(a_0, a_1)+e(a_1, a_2) + \cdots +e(a_{n-1}, a_n)$$

is the smallest possible.

There are various procedures for finding a solution. A most elegant one follows.[1] We mark the initial vertex a_0 with the number $M(a_0)=0$; each of the other vertices v in G is marked by a number $M(b)$ exceeding all distance numbers $e(a, b)$; one may even write $M(b)=\infty$. We now begin a repeated reduction of these numbers. When the pair of vertices u and v connected by an edge (u, v) have the associated numbers $M(u)$ and $M(v)$ with

$$e(u, v)<M(v)-M(u)$$

we replace $M(v)$ by the smaller number

$$M_1(v)=M(u)+e(u, v)$$

At the same time, we indicated at v that the reduction was performed from the

[1] See for instance M. Beckman, C. B. McGuire, and C. B. Winsten, *Studies in the Economics of Transportation* (New Haven: Yale University Press, 1956) and L. R. Ford, *Network Flow Theory*, Rand Corp. Publ. P-923 (1956).

vertex u. The first step in this reduction will evidently take place at one of the vertices v_1 which can be reached by an edge from a_0 and the new value at v_1 will be the distance $e(a_0, v_1)$.

Since the graph is finite, the reduction process must eventually come to a halt. At the vertex $b_0 = a_n$ there will be a neighboring vertex a_{n-1} used in the last reduction at a_n, similarly at a_{n-1} there is a last reduction with respect to some a_{n-2}, and so on. Correspondingly, one has a decreasing sequence of index numbers

$$\begin{aligned} (3) \qquad M(a_n) &= M(a_{n-1}) + e(a_n, a_{n-1}) \\ M(a_{n-1}) &= M(a_{n-2}) + e(a_{n-1}, a_{n-2}) \\ &\cdots\cdots\cdots \end{aligned}$$

The corresponding path can only terminate at a_0, for at all other vertices there has been a reduction at some time. Thus we have determined a path P as in (1) connecting a_0 and b_0. By adding the equations (3), one sees that the length of P is the reduced index value at b_0

$$e(P) = M(b_0)$$

It remains to show that the path P we have arrived at gives the smallest possible value for $e(P)$. To prove this, let

$$Q = (a_0, b_1)(b_1, b_2) \cdots (b_{m-1}, b_0)$$

be another connecting itinerary between a_0 and b_0. Since there are no more possible reductions, one must have

$$\begin{aligned} M(b_0) &\leqq M(b_{m-1}) + e(b_0, b_{m-1}) \\ M(b_{m-1}) &\leqq M(b_{m-2}) + e(b_{m-1}, b_{m-2}) \\ &\cdots\cdots\cdots \end{aligned}$$

By adding these inequalities one finds

$$M(b_0) \leqq e(b_0, b_{m-1}) + e(b_{m-1}, b_{m-2}) + \cdots$$

or

$$e(Q) \geqq e(P) \text{ as desired.}$$

This ingenious method suffers from a serious drawback: for most applications it is highly impractical. In particular, it requires the reduction of the entire graph. The situation is similar to that in the labyrinth problem which we discussed in a previous article.[2] Here there were elegant procedures to cover all passageways in the maze to the most remote crevices, yet this was not required to achieve the essential object of the search, to find the entrance from which one had come.

A method related to the one we used in the labyrinths can also be applied to determine the shortest itinerary. Beginning at the initial vertex a_0 in the graph G, one proceeds to all vertices v_1 which can be reached by a single edge. At each v_1 one writes the distance from a_0, the shortest if there should be several edges to v_1. From each v_1 one proceeds to the vertices v_2 which can be reached from a vertex v_1 by a single edge. For each v_2 one notes the shortest distance and the corresponding path from a_0. One then continues to the vertices v_3 connected by an edge not previously considered to a vertex v_2 and writes shortest distance and path.

In this manner one must eventually, say in m_1 steps, reach the destination $v_{m_1} = b_0$ in a path P_1 of length $d_1(b_0)$. One can then discard all other shortest paths to vertices v_{m_1} with

$$d_1(v_{m_1}) \geqq d_1(b_0)$$

since by continuation they cannot give any shorter path to b_0. The others are continued along new edges as before, but they are discontinued when their length exceeds $d_1(b_0)$. One may reach b_0 on another path P_2 with $m_2 > m_1$ edges but a distance

$$d_2(b_0) < d_1(b_0)$$

One then discards P_1 and all paths to vertices v_{m_2} with

$$d_2(v_{m_2}) > d_2(b_0)$$

Since all distances increase, this construction must cease after a certain number of steps, and the shortest path $P(a_0, b_0)$ is obtained.

This last construction is particularly effective when one wants to determine the shortest or fastest or cheapest route to all

[2] O. Ore, "An Excursion into Labyrinths," THE MATHEMATICS TEACHER, LII (May 1959), 367–70.

localities within a region surrounding some distribution center. Then each driver can be given ready-made instructions for the most economical way of getting to his destination. The situation is much more complicated when the driver is to visit several places on the same trip. If there are only two places, a_1 and a_2, he can always take a triangular route a_0, a_1, a_2, a_0 on the shortest connecting roads. But if there are more destinations, no such easy rule can be given. The reader may try to analyze the alternatives when there are three destinations a_1, a_2, a_3 from a_0.

The general problem of finding the shortest route passing through n localities with given distances

$$a_1, a_2, \cdots, a_n$$

and returning to the point of departure a_0 is known as the *Traveling Salesman Problem.* No effective algorism for determining the solution is known in spite of the fact that it is a most important problem for various phases of operations research. In its original form, it deals with a salesman assigned to a territory consisting of certain cities to be visited. Naturally he wishes to return to his home office after as little travel as possible. But the problem has other, more important applications to the planning for an effective use of personnel, rolling stock, planes, etc. A large scale example of the Traveling Salesman Problem has been carried through by Dantzig, Fulkerson, and Johnson in determining the shortest air path connecting the various capitals of the states in the U.S.A.[3]

[3] G. Dantzig, R. Fulkerson, S. Johnson, "Solution of a Large Scale Traveling Salesman Problem," *Journal of the Operations Research Society of America*, II (1954), 393–410. See also M. M. Flood, "The Traveling Salesman Problem," *ibid.*, IV (1956), 61–75.

What's new?

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A two-and-one-half-place logarithm table

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Approximating log x *by finding integers* m *and* n *such that* x^m *is approximately equal to* 10^n.

THE CLASSICAL, BUT too little-known, method for obtaining two-place logarithms of the integers between 1 and 10 has recently been incorporated in two texts.[1] In the present paper we elaborate on this method and set up a two-place table of all integers between 1 and 100. In fact, our table, without being accurate to three decimals, is more accurate than a two-place table and may be said to be correct to two-and-a-half places. Students find this method stimulating and interesting. This method clearly illustrates for the student the relationship between exponents and logarithms, and shows how to deal with errors inherent in all log tables.

Besides the *primary key entries* in the table, that is, the logarithms of the integers between 1 and 10, we determine approximate values of the logarithms of a few numbers between 10 and 100, which we call *secondary key entries*. All other entries are computed by interpolation.

THE PRIMARY KEY ENTRIES

The method is based on the fact that 2^{10} (that is, 1024) is approximately equal to 10^3 (that is, 1000). We will write "$\sim$" for "approximately equal."

$2^{10} \sim 10^3$ implies $2 \sim 10^{3/10} = 10^{.3}$ and hence $\log 2 \sim .3$. While 1024 exceeds 1000 by almost $2\frac{1}{2}\%$, fortunately log 2 exceeds .3 by less[2] than .00104 (or about $\frac{1}{3}\%$). More precisely, to five decimals, $\log 2 = .30103$; therefore .30 is accurate to two decimals.

From $\log 2 \sim .30^+$ it follows that $\log 4 \sim .60^+$ and $\log 8 \sim .90^+$. Moreover,

$$\log 5 = \log 10 - \log 2 \sim 1 - .30^+ \sim .70^-.$$

The raised + (or −) after an approximate value indicates that that number has to be increased (or decreased) to yield the actual logarithm. All four logarithms mentioned are accurate to two places. Even log 8 exceeds .90 by less than .0031.

The situation concerning log 3 and log 9 is somewhat less favorable. 3^4 (that is, 81) exceeds 80 by $1\frac{1}{4}\%$. Hence

$$4 \log 3 \sim \log 8 + \log 10 \sim 1.9^+$$

and

$$\log 3 \sim .475^+.$$

On the other hand, 3^9 (that is, 19683) is close to 20000 or $2 \cdot 10^4$, wherefore[3]

$$9 \log 3 \sim \log 2 + \log 10^4 \sim 4.3^+$$

and

$$\log 3 \sim \frac{4.3}{9} = .4\dot{7}.$$

[1] Karl Menger, *Calculus. A Modern Approach* (Boston: Ginn and Co., 1955), p. 23, and *Basic Concepts of Mathematics. A Companion to Current Textbooks. Part I. Algebra* (Chicago: Illinois Institute of Technology Bookstore, 1957), p. 49, 50.

This paper has developed from a problem formulated to the author in 1957 by Professor Menger in his course Mathematics 400, *Basic Concepts of Mathematics*, which for the past two years has been offered in N.S.F. sponsored In-Service Institutes for High School Teachers at the Illinois Institute of Technology.

[2] This can be proved by calculus. See Menger, *Calculus. A Modern Approach, loc. cit.*, p. 288.

[3] Since 9 log 3 is less than 4 + log 2 but log 2 > .30, the reasoning above permits only the inference that $\log 3 \sim .4\dot{7}$ and not that $\log 3 < .4\dot{7}$. But, as one can prove by calculus, this inequality is valid.

Here, $.4\dot{7}$ is an abbreviation of the periodic decimal fraction $.477 \cdots 7 \cdots$ To two places, $\log 3 = .48$. Since, to five places, $\log 3 = .47712$, the two-place approximation .48 is closer to log 3 than .90 is to log 8.

In computing log 6 as $\log 2 + \log 3 \sim .30^{+} + .48^{-} \sim .78$, the two inaccuracies partly cancel one another, and .78 actually exceeds log 6 by less than .002. If we computed log 9 as 2 log 3, however, we would double the inaccuracy committed by setting $\log 3 \sim .48$. We therefore will double $.4\dot{7}$ rather than .48. The number $.9\dot{5}$ thus obtained is about midway between .95 and .96, therefore we will use .955 for the table. The introduction of midpoints between numbers differing by .01 will result in a table that is more accurate than a mere two-place table.

Such a midpoint is particularly appropriate for log 7. Since 7^4 (that is, 2401) is close[4] to 2400 (which is $3 \cdot 8 \cdot 10^2$) it follows that

$$4 \log 7 \sim \log 3 + \log 8 + 2 \sim 3.38$$

and

$$\log 7 \sim .845.$$

This number is an approximation for log 7 to three decimals.

The actual entries in our table will be, as is customary, the mantissas of the logarithms. Those for the integers between 1 and 10, the primary key entries, are

1	2	3	4	5	6	7	8	9	10
00	30^{+}	$^{*}48^{-}$	60^{+}	70	78	84_5	$^{*}90^{+}$	95_5	00

To lessen confusion arising from having three-digit numbers in what is essentially a two-place table, we lower the 5 in the third place of the mantissas. The logarithms of 3 and 8, marked with an asterisk, will be slightly corrected in Section 3.

The secondary key entries

By means of the primary key entries we next determine the logarithms of a few numbers that are the products of integers between 1 and 10.

	Mantissa
$\log 12 = \log 3 + \log 4 \sim 1.08$	08
$\log 14 = \log 2 + \log 7 \sim 1.145$	14_5
$\log 25 = 2 \log 5 \sim 1.40$	40
$\log 28 = \log 4 + \log 7 \sim 1.445$	44_5
$\log 35 = \log 5 + \log 7 \sim 1.545$	54_5
$\log 36 = \log 4 + \log 9 \sim 1.555$	55_5
$\log 45 = \log 5 + \log 9 \sim 1.655$	65_5
$\log 48 = \log 4 + \log 12 \sim 1.68$	68

It will be noted that we have not determined log 16 as 4 log 2; the reason being that we avoid quadrupling the inaccuracy committed by setting $\log 2 \sim .30$. For a similar reason we have refrained from determining log 27 as 3 log 3.

Since $\log 10 = 1.00$ and $\log 12 \sim 1.08$, it follows from the interpolation law (to be discussed in Section 3) that log 11 cannot be less than 1.04. On the other hand,

$$11 < \frac{100}{9} \text{ implies that}$$

$$\log 11 < 2 - \log 9 \sim 1.045.$$

We therefore set $\log 11 \sim 1.04$. (Actually, to five decimals, $\log 11 = 1.04139$.)

In possession of log 11, we can determine the logarithms of 22, 33, 55, 66, and 77. The mantissas exceed by .04 those of 2, 3, 5, 6, and 7, respectively.

At this stage, the tentative two-place table looks like accompanying Table 1, on page 185.

The last column has been given to facilitate interpolation.

Interpolation[5]

From

$$\frac{a}{a-1} > \frac{a+1}{a} \text{ for } a > 1,$$

[4] This was pointed out to the author by Mr. D. M. Hyman.

[5] This section was suggested by and written with the assistance of Professor Karl Menger at Illinois Institute of Technology, whose time was available through a grant of the Carnegie Corporation of New York for the development of Menger's new approach to the teaching of intermediate mathematics.

TABLE 1—Tentative two-place table

	0	1	2	3	4	5	6	7	8	9	
1	00	04	08		14_5						30^+
2	30^+		34			40			44_5		$*48^-$
3	$*48^-$			52		54_5	55_5				60^+
4	60^+					65_5			68		70
5	70					74					78
6	78						82				84_5
7	84_5							88_5			$*90^+$
8	$*90^+$										95_5
9	95_5										100

it follows that

$$\log a - \log (a-1) > \log (a+1) - \log a$$
$$\text{for each integer } a > 1.$$

Of course this inequality is valid for sufficiently accurate approximate logarithms of the three consecutive integers $a-1$, a, $a+1$; that is to say, in a log table of sufficiently many places, the difference between the entries for a and $a-1$ exceeds the difference between the entries for $a+1$ and a. In a table that is not sufficiently accurate, however, there may be exceptions to this rule of decreasing differences. For instance, there are exceptions in a one-place table of the integers between 1 and 10, which (according to the preceding two-place table) reads as shown in the first of the two rows of bracketed numbers below.

Not only have five pairs of consecutive entries the same difference, namely 1, but the difference 0 between the entries for 6 and 7 is followed by the greater difference 1 between the entries for 7 and 8.

Five-place tables for the integers from 1 to 10000 abound in exceptions to the rule of decreasing differences. For instance, on the page devoted to the 500 numbers from 4500 to 5000, only three differences between consecutive entries occur: 10, 9, and 8. Consequently, many consecutive differences are equal. It cannot even be claimed that all the differences 9 occur after all the differences 10. For instance, the entries for the numbers from 4500 to 4509 read as shown in the second row below.

Naturally, we shall face the same dif-

Number:	1	2	3	4	5	6	7	8	9	10
Mantissa:	0	3	5	6	7	8	8	9	0	0
Difference:		3	2	1	1	1	0	1	1	0

Number:	450	0	1	2	3	4	5	6	7	8	9
Mantissa:	65	321	331	341	350	360	369	379	389	398	408
Difference:			10	10	9	10	9	10	10	9	10

ficulties in setting up a two-place table for the integers between 1 and 100. All that can be done is *to avoid, as far as possible, increasing differences between consecutive entries.*

In the first row of our tentative table, the difference between the entries 08 and 14_5 is 06_5. The greater part should go into the difference between the entries for 12 and 13, the smaller part into the difference between 13 and 14. This yields the entry 11_5 for 13.

The entries for 14 and 20 differ by 15_5. The best partition of this difference into six differences is

$$15_5=03+03+02_5+02_5+02_5+02.$$

The resulting entries

for	14	15	16	17	18	19	20
are	14_5	17_5	20_5	23	25_5	28	30.

The entry 25_5 for 18 actually is the mantissa of $\log 2+\log 9 \sim 1.25_5$.

In the second row, interpolation suggests the entry 32 for 21 as well as 36 for 23 and 38 for 24. Indeed, $1.38 \sim \log 3+\log 8$.

The best partition of the difference between 40 and 48^- in five differences is

$$08=01_5+01_5+01_5+01_5 02.$$

This yields the following entries:

41_5 for 26, which is confirmed by the fact that $1.41_5 \sim \log 2+\log 13$;

43 for 27;

44_5 for 28, which is confirmed by $1.44_5 \sim \log 4+\log 7$;

46 for 29.

That the difference between the entries for 29 and 30 exceeds the preceding difference is mitigated by the fact that $\log 3<.48$.

For the same reason, the difference between 48 and 52 in the third row should be partitioned

$$04=01+01_5+01_5$$

which yields the entries 49 for 31, and 50_5 for 32.

The nine entries for the numbers from 25 to 33 thus read

25	26	27	28	29	30	31	32	33
40	41_5	43	44_5	46	48	49	50_5	52

with differences:

$$01_5 \quad 01_5 \quad 01_5 \quad 01_5 \quad 02 \quad 01 \quad 01_5 \quad 01_5.$$

Inspection of the differences suggests a correction of log 3 from $.48^-$ to .475. And indeed, since, to five decimals, $\log 3 =.47712$, the logarithm of 3 is slightly closer to .475 than to .48. The corrected entry for log 3 is also in agreement with the entries 32 for 21, and 43 for 27, since

$$\log 21=\log 3+\log 7 \sim .475+.845 \sim 1.32;$$

$$\log 27=\log 3+\log 9 \sim .475+.955 \sim 1.43.$$

The proper entry for 34 is 53, which is confirmed by $1.53 \sim \log 2+\log 17$.

The best partition of the difference between 55_5 and 60^+ is

$$04_5=01_5+01+01+01,$$

which yields the entries 57, 58, and 59 for 37, 38, and 39, respectively. The entries for 38 and 39 are in agreement with $38=2\cdot 19$ and $39=3\cdot 13$ if one uses the corrected log 3.

In the fourth line, the best interpolation between 60^+ and 65_5 is

$$60^+ \quad 61_5 \quad 62_5 \quad 63_5 \quad 64_5 \quad 65_5.$$

Reasonable, if not straightforward, interpolations between 65_5 and 68, between 68 and 70^-, and between 70^- and 74 are

$$65_5 \quad 66_5 \quad 67 \quad 68; \qquad 68 \quad 69 \quad 70;$$

and

$$70^- \quad 71 \quad 71_5 \quad 72_5 \quad 73 \quad 74.$$

The entries for 49, 52, and 54 are in agreement with

$$1.69 \sim 2 \log 7, \quad 1.71_5 \sim \log 2+\log 26,$$

$$1.73 \sim \log 2+\log 27.$$

Since

$$\log 56=\log 2+\log 28 \sim .30+1.445 \sim 1.745$$

$$\log 57=\log 3+\log 19 \sim .475+1.28 \sim 1.755$$

$$\log 58 = \log 2 + \log 29 \sim .30 + 1.46 \sim 1.76,$$

we enter 74_5, 75_5, and 76 for 56, 57, and 58, respectively, and of course 77 for 59.

Since

$$\log 62 = \log 2 + \log 31 \sim .30 + 1.49 \sim 1.79$$

$$\log 63 = \log 7 + \log 9 \sim .845 + .955 \sim 1.80$$

$$\log 64 = \log 2 + \log 32 \sim .30 + 1.505 \sim 1.805$$

$$\log 65 = \log 5 + \log 13 \sim .70 + 1.115 \sim 1.815,$$

we enter 79, 80, 80_5, and 81_5 for 62, 63, 64, and 65, respectively.

Since

$$\log 69 = \log 3 + \log 23 \sim .475 + 1.36 \sim 1.835,$$

we enter 83_5 for 69, which entails the entries 82, 82_5, and 83 for 66, 67, and 68. The latter is in agreement with $.83 \sim \log 2 + \log 34$.

$$\log 72 = \log 4 + \log 18 \sim .60 + 1.255 \sim 1.855$$

$$\log 75 = \log 3 + \log 25 \sim .475 + 1.40 \sim 1.875$$

$$\log 78 = \log 2 + \log 39 \sim .30 + 1.59 \sim 1.89.$$

These results, in conjunction with reasonable interpolations, account for the following entries in the seventh line of the table except for the entry for 79:

$$84_5 \quad 85 \quad 85_5 \quad 86_5 \quad 87 \quad 87_5 \quad 88 \quad 88_5 \quad 89.$$

In the eighth row, plus the entry for 79, the difference 95_5–89, that is, 06_5, has to be partitioned in twelve differences, which makes eleven of them 00_5 and one of them 01. Since our entry for 80 is 90^+, this suggests we let $\log 80 \sim 1.905$ and $\log 79 \sim 1.895$. This leaves ten differences of $.00_5$ for the eighth row. The entries in the table are, therefore,

$$90_5 \quad 91 \quad 91_5 \quad 92 \quad 92_5 \quad 93 \quad 93_5 \quad 94 \quad 94_5 \quad 95.$$

These entries are in agreement with

$$1.915 \sim \log 2 + \log 41,\ 1.925 \sim \log 2 + \log 42,$$
$$1.945 \sim \log 8 + \log 11.$$

In the last row, the difference 100–95_5, that is, 04_5, has to be partitioned in ten differences. This means that nine of them are 00_5, while one of them is 00. Choosing the two equal consecutive entries in the middle of the line one obtains the entries:

$$95_5 \quad 96 \quad 96_5 \quad 97 \quad 97_5 \quad 98 \quad 98 \quad 98_5 \quad 99 \quad 99_5.$$

The entries for 92, 95, 96, 98, and 99 are in agreement with

$$1.965 \sim \log 2 + \log 46$$
$$1.98 \sim \log 5 + \log 19 \sim \log 3 + \log 32$$
$$1.99 \sim \log 2 + \log 49$$
$$1.995 \sim \log 9 + \log 11.$$

Summarizing, we can set up Table 2.

TABLE 2—Two-and-one-half-place logarithm table

	0	1	2	3	4	5	6	7	8	9
1	00	04	08	11_5	14_5	17_5	20_5	23	25_5	28
2	30	32	34	36	38	40	41_5	43	44_5	46
3	47_5	49	50_5	52	53	54_5	55_5	57	58	59
4	60	61_5	62_5	63_5	64_5	65_5	66_5	67	68	69
5	70	71	71_5	72_5	73	74	74_5	75_5	76	77
6	78	78_5	79	80	80_5	81_5	82	82_5	83	83_5
7	84_5	85	85_5	86_5	87	87_5	88	88_5	89	89_5
8	90_5	91	91_5	92	92_5	93	93_5	94	94_5	95
9	95_5	96	96_5	97	97_5	98	98	98_5	99	99_5

REMARKS

Comparison with a four-place table reveals that few entries in our $2\frac{1}{2}$-place table are inaccurate. Since

$$\log 56, \quad \log 58, \quad \log 69$$

have the four-place mantissas

$$7482, \qquad 7634, \qquad 8388,$$

the entries in our table should be

$$75, \quad 76_5, \quad 84 \quad \text{and not} \quad 74_5, \quad 76, \quad 83_5.$$

The logarithms of the numbers from 40 to 47 have the four-place mantissas

$$\begin{array}{cccc} 6021 & 6128 & 6232 & 6335 \\ 6435 & 6532 & 6628 & 6721. \end{array}$$

The last two digits of these numbers are so close to 25 that it is impossible to devise correct $2\frac{1}{2}$-place mantissas without a more penetrating analysis, and the accuracy of the entries in the fourth row of our table is coincidental. A slightly different, and rather plausible, reasoning would suggest to partition the difference 70–60 of $\log 5 - \log 4 = .70 - .60$ in ten equal differences, .01 each. This would have led to the following fourth row:

$$60\ 61\ 62\ 63\ 64\ 65\ 66\ 67\ 68\ 69.$$

And indeed this row would be only slightly inferior to that in our table.

A related difficulty concerns multiplication. The worst shortcoming of a $2\frac{1}{2}$-place table is the fact that

$$\log 3 = .475 \quad \text{and} \quad 2 \log 3 = .95,$$
$$\text{whereas } \log 9 = .955.$$

But since, to four places, $\log 3 = .4771$ and $\log 9 = .9542$, the discrepancy is inevitable. Similar difficulties exist in all tables. For instance, in a four-place table

$$\log 4 = .6021 \quad \text{and} \quad 2 \log 4 = 1.2042,$$
$$\text{whereas } \log 16 = 1.2041.$$

Our table is also in minor disagreement with the multiplications

$$8 \cdot 9, \quad 2 \cdot 21, \quad 2 \cdot 23, \quad 2 \cdot 28$$

and a few others.

Have you read?

SCHUSTER, CARL N. "Locating the Decimal Point," *Scripta Mathematica*, June 1959, pp. 149–53.

You may have tried many methods of teaching your students to locate the decimal point when solving problems. This author gives you a best method. His basis is the writing of all numbers in standard notation, that is as powers of ten, positive or negative. In this way significant digits are known, decimal points can be located correctly when the slide rule or computing machines are used, and the student obtains an understanding of the quantitative nature of number. This method brings logarithms to the front and provides an easy introduction to them. You should read this article, then decide if this is the way to keep the wandering decimal point in place.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

STRINGFELLOW, EMMA L. "Number Systems," *School Science and Mathematics*, October 1959, pp. 557–60.

Today there are many articles and ideas on number systems. This is a short but to the point article on the same. The author recommends the building of new systems to make the decimal system more meaningful. The steps she would follow are: (1) determine how consecutive numbers are built in the decimal system; (2) build the new system and its addition tables; (3) find a formula for writing a numeral of any value in the new system. She proceeds to show examples of a 5 system and a 3 system, as well as examples of where other systems are now used. You may want to give your students the practice problems at the end of the article. I am sure they would enjoy them.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

The law of contraposition

JOHN A. TIERNEY, *United States Naval Academy, Annapolis, Maryland.*

Some examples are given to show how students can make use of the law of contraposition in the solution of some problems.

MOST STUDENTS WHO have studied demonstrative geometry are familiar with the converse of a proposition, but few are acquainted with its contrapositive. Let us assume that a given proposition has been stated in the form:

If A is true, then B is true.

A is called the hypothesis of the proposition and B its conclusion. The converse of the proposition is obtained by interchanging the hypothesis and the conclusion, and may be either true or false. If we deny or negate the hypothesis and the conclusion and also interchange them, we obtain the contrapositive of the proposition. The contrapositive of our given proposition is:

If B is not true, then A is not true.

An example of such a contrapositive is:

Proposition: If I live in Annapolis, then I live in Maryland.

Contrapositive: If I do not live in Maryland, then I do not live in Annapolis.

Another example is:

Proposition: If the roots of $ax^2+bx+c=0$ are equal, then $b^2-4ac=0$.

Contrapositive: If $b^2-4ac\neq 0$, then the roots of $ax^2+bx+c=0$ are not equal.

In texts on logic, the standard method of a truth table is employed to prove that a proposition and its contrapositive are both true or both false. This result is known as the Law of Contraposition. If students of mathematics were familiar with this law they would find it very useful in analyzing both mathematical and nonmathematical propositions. A statement of the law could be given, and the students could construct illustrations.

It is interesting to note that the familiar method of indirect proof is an application of the Law of Contraposition. Suppose we wish to prove a theorem of the form "If p is true, then q is true." We assume that q is false and show as a consequence that p is false. We then conclude that q is true since we have arrived at an obvious contradiction. Actually we have demonstrated that $\sim q \rightarrow \sim p$ (Read: not q implies not p.), and hence its contrapositive $p \rightarrow q$ must also be true. To illustrate, suppose we wish to prove the following theorem from plane geometry:

If two distinct lines are perpendicular to the same line, then they are parallel.

We prove instead the following:

If two distinct lines are not parallel, then they are not perpendicular to the same line.

Having established the second theorem, we know the first to be true since they are contrapositive to one another.

We now consider an illustration from algebra. Let us assume that in studying the theory of equations we have established the following theorems concerning a polynomial $f(x)$:

1. If $f(r)=0$, then $(x-r)$ is a factor of $f(x)$.
2. If $(x-r)$ is a factor of $f(x)$, then $f(r)=0$.

Theorem 1 is called the Factor Theorem, and Theorem 2 is its converse. Now sup-

pose we give a student the following problem: Is $x-1$ a factor of $x^{93}-1$?

If the student is competent, he will recognize that $f(1)=0$, thus satisfying the hypothesis of Theorem 1, and that as a consequence the conclusion of Theorem 1 follows, namely that $(x-1)$ is a factor of $x^{93}-1$.

Now give the same student this problem: Is $x+1$ a factor of $x^{93}-1$?

Even a competent student will find this much more difficult than the first problem. The trouble is that he is unable to satisfy the hypothesis of either Theorem 1 or Theorem 2. The problem is easy if we write the contrapositive of Theorem 2:

If $f(r)\neq 0$, then $(x-r)$ is not a factor of $f(x)$.

Since $f(-1)=-2\neq 0$, $x-(-1)=x+1$ is not a factor of $x^{93}-1$.

Another application is afforded by the study of an equation of a locus, defined as follows:

$f(x, y)=0$ is an equation of a locus if it is satisfied by the co-ordinates of every point of the locus and by the co-ordinates of no other point.

This definition includes the following two statements:

1. If $P(x, y)$ is on the locus, its co-ordinates satisfy $f(x, y)=0$.
2. If $Q(x, y)$ is not on the locus, its co-ordinates do not satisfy $f(x, y)=0$.

This is very confusing to the student because most texts, in deriving an equation of a locus, show that the first statement holds and then verify the contrapositive of the second statement, namely:

If the co-ordinates of $Q(x, y)$ satisfy $f(x, y)=0$, then $Q(x, y)$ is on the locus.

For example, it is shown that if $P(x, y)$ is (with axes appropriately chosen) any point of an ellipse, then

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Next it is demonstrated that if

$$\frac{x_0{}^2}{a^2}+\frac{y_0{}^2}{b^2}=1,$$

then the point (x_0, y_0) is on the ellipse.

This development will be clearer to the student once he understands that the second statement and its contrapositive are equivalent statements. A discussion of the following two problems is also instructive.

1. Is $x^2+y^2-1=0$ an equation of the semicircle through $(-1, 0)$, $(0, 1)$, and $(1, 0)$?
2. Is $y-\sqrt{1-x^2}=0$ an equation of the unit circle with center at the origin?

We next consider an example from the calculus. In the study of infinite series we prove the following:

If $\sum_{n=1}^{\infty} u_n$ converges, then $\lim_{n\to\infty} u_n=0$.

We then ask the student to demonstrate that the series

$$\sum_{n=1}^{\infty}\frac{n}{n+1}$$

diverges. This puzzles him because the hypothesis of the statement given involves a series which converges and his problem concerns a series which supposedly does not converge. The problem is simplified if he realizes that the contrapositive is also true, namely:

If $\lim_{n\to\infty} u_n\neq 0$, then $\sum_{n=1}^{\infty} u_n$ does not converge; that is, $\sum_{n=1}^{\infty} u_n$ diverges.

These are but a few of the instances in which an acquaintance with the Law of Contraposition would be helpful to the student.

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Numbers, numerals, and last digits[1]

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Since '1/4', '.25', '.250', and '.24$\dot{9}$' $\cdots$ are all names for the same number, is it correct to speak of "the last digit of a number"?

SINCE THE TURN of the century, under the impetus of the work of the German logician G. Frege,[2] logicians have stressed the difference between numbers and numerals, i.e., symbols for numbers. Recognition of this distinction has finally entered the realm of secondary education, and we should continually caution our pupils about the difference between a number and its numeral. Are we teachers fully aware of all the differences between these two concepts? The purpose of this article is to indicate one instance where a property, commonly attributed to numbers, is, in fact, a property of the numerals that represent those numbers.

There seems to be no problem in ascertaining "the last digit of the number .2498." Upon inspection of the numeral, one would say '8'. Similarly, the last digits in '.249', '.2499', and '.24999' would be determined to be '9', where the nonterminating periodic decimal '.24999 . . . ' (hereafter symbolized by '.24$\dot{9}$') has no last digit. Similarly, the last digits in '.25', '250', and '.2500' are '5', '0', and '0' respectively, while '25$\dot{0}$' has no last digit. But the numerals '.25', '250', '.2500', and '.25$\dot{0}$' represent the same number which also has the numeral '$\frac{1}{4}$'. Moreover $.24\dot{9} = \frac{1}{4}$. Therefore, even '.24$\dot{9}$' is another numeral for the same number. Hence one cannot speak about the last digit of a number. One can only ascertain the last digit of a specific decimal numeral for that number, various numerals having various last digits or even no last digit at all.

In the same fashion, one sees that the third digits in the numerals '.24$\dot{9}$', '.250', '.2500', and '.25$\dot{0}$' are, respectively, '9', '0', '0', and '0', while '.25' has no third digit. Thus, the third digits of the numerals for .250 are also not uniquely determined. Hence, in certain cases, one cannot even speak about the third decimal digit.

It should be noted that this ambiguity concerning digits other than the last arises only when a number has at least one terminating decimal numeral. Thus no ambiguity arises when one asks for the fourth decimal digit of the decimal symbol for π or $\sqrt{2}$ or .3434 $\cdots$ (.$\dot{3}\dot{4}$). The only way this problem can be resolved for numbers with at least one terminating decimal numeral is by rephrasing the problem. We may properly seek the third and last digits in the various decimal numerals for $\frac{1}{4}$. The answers are: '0' and '0' in '.250'; '0' and '0' in '.2500'; '0' and none in '.25$\dot{0}$'; '9' and none in '.24$\dot{9}$'; and none and none in '.25.' Therefore, the digits depend upon the symbol, or numeral, one chooses to use for that number.

An interesting aspect of the property of numerals here discussed is the answer to the question, "What is known about a number if the first three digits in some of its decimal numerals are in order '4', '6', and '3'?" Since the smallest number designated by these three digits in this order is $.463\dot{0}$ and the largest is $.463\dot{9} = .464$, any number, c, with this property satisfies the inequality $.463 \leq c \leq .464$; that is, the number is neither less than .463 nor greater than .464, but it may be equal to either of these numbers.

[1] This article was written as a result of investigations stimulated by the In-Service Institute for High School Teachers conducted by Dr. Karl Menger and sponsored by the National Science Foundation at Illinois Institute of Technology, Chicago, Illinois.

[2] Karl Menger, *The Basic Concepts of Mathematics* (Chicago: The Bookstore, Illinois Institute of Technology, 1957), Chap. I.

Edited by Howard Eves, University of Maine, Orono, Maine

Monsieur Fermat

by Sister Marie Stephen, O.P., Rosary College, River Forest, Illinois

Very often one finds a little town producing a great genius. The village of Beaumont de Lomagne on the Garonne and Tarne Rivers in the province of Gascogny in southern France became on an August morning the birthplace of just such an incipient genius. Dominic Fermat, an esteemed member of the tiny community, and his wife, Clare de Long, took their young son to the parish church on August 20, 1601 to be christened Pierre de Fermat, a name which was destined to live long past the child.

The little boy was brought up like most other little boys. We have no stories of his childhood genius because the earliest details of his life remain obscure. His mother was of a very learned family of Parliamentarians. She probably taught him his first lessons in his father's leather shop. They were lessons of kindness and humility which so characterized his nature. From his father, who besides being the town leather merchant was second councilor of the village, the boy gained his knowledge of the law. From his family in general he must have become permeated with that peace which he loved all his life. Indeed the boy was very unworldly, so much so that he stands out in later life as a naive and peaceful figure against a tumultuous political and intellectual background.

The family later moved to Toulouse, where Pierre received the last part of his education at the famous and ancient university of that city. Here he lived most of his life in the midst of medieval learning.

When he was thirty years old he began his own family life. Louise de Long, a cousin of his mother, became his wife on the first of June, 1631. The story of his married life is a reflection of the principles with which he was so thoroughly imbued in his father's home. And Pierre de Fermat's own family became just as much a monument to that Christian gentleman as did his works. Samuel-Clément, born in 1632, was destined to become his father's successor in the Parliament, and, as a writer and classicist, the chief publisher of his father's works after that eminent mathematician's death. The second son, Jean, became a priest and later Arch Deacon of Fimarens, while Catherine and Louise, the youngest daughters, became members of religious orders. The eldest daughter, Clare, was happily married, and her grandson, Jean Gailhard, succeeded Samuel's son, Jean François, as Councilor in the Parliament. It is evident that no matter how full Pierre Fermat's life was, he did not lose sight of his children and gave them the same training he himself had had.

We learn from Fermat's letters that although his many avocations held first place in his heart, his legal business nevertheless was always executed with scrupu-

lous fidelity. The year 1634 saw him nominated Councilor of the Parliament of Toulouse. This Council had been made strictly a judicial affair by Cardinal Richelieu and divested of any political authority. Monsieur Fermat became known throughout France, and in fact all Europe, as "le célèbre Conseiller de Toulouse." This we learn from his contemporaries, for, as in his personal life so in his profession, he had a great aversion to publicity—as much a mark of distinction in that day as in our own.

The man was an artist and a scholar. After his work was finished one can imagine him devoting half his time to the instruction of his children and the other half to scholarly pursuits. He was interested in music, poetry, and the classics. He found time to correspond with Balzac, Houet, Chapelain; to read and criticize their works; to philosophize with them and against them, always in a kind, frank manner. He recorded in Latin the local customs and life of his native village of Beaumont de Lomagne. He made a thorough study and wrote commentaries on Polyaenus, Synesius, Theon of Smyrna, and many other ancient Latin writers. We find in his works several beautiful poems in Latin, among them the "Cede Deo," a fine piece of philosophy dressed in the most mellifluous Latin verse. It seems almost impossible to the average human being that a man so active in all these fields could possibly find time for any more. Yet the greatest of his hobbies, the closest to his heart, that for which he will never be forgotten, was his devotion to the queen of sciences. In a century when names such as Newton, Leibniz, Bernoulli, Huygens, Torricelli, Roberval, Pascal, Descartes, Wallis, Cavalieri, Oughtred, Carcavi, Mersenne, and many others were becoming famous, the name of Pierre de Fermat dominated the whole circle—not in rank of publicity, but in genius and universality of knowledge. Fermat knew all these men; they were his friends and correspondents. It was through his letters to them that the world has a record of his many valuable contributions to mathematics. He was loved and respected by every one of these great men. Even Descartes realized his superiority. They sought his advice on their inventions and were the recipients of his vast knowledge. In a century when Leibniz and Newton contended for the priority of the invention of the Calculus, and the scientists of the day were jealously guarding their secrets for fear of plagiarism, Fermat was giving generously and unassumingly to the world of savants great and unheralded discoveries, with no desire for praise except that one appreciate the "beauty" of some particular theorem.

Nine years before Descartes published his work on Analytic Geometry, the principal concepts of that analysis appeared in Fermat's unpublished commentary on "Lieux plans d'Apollonius." The keynote of the Differential Calculus appeared from his pen before Leibniz and Newton even thought of contesting it. We see the beginnings of the Calculus of Probability in his correspondence with Pascal long before Jacques Bernoulli got hold of it, while the great and new study of Number Theory appeared as marginal notes on Bachet's edition of Diophantus and in letters to many friends. Monsieur Baccou writes,[1] "Perhaps unique in the history of science and letters is the case in which this genial soul shows itself in Geometry, Algebra, and finally in the Theory of Numbers —one of the great modern sciences—and has not shown any of his works in public."

Of over 3280 mathematical manuscripts written during his lifetime, Fermat published only one, and that just five years before his death, under the initials M.P.E.A.S. As deep a philosopher and artist as he was a scientist, he knew the futility of worldly honors and loved mathematics for its sake alone. On one occasion Descartes, highly temperamental

[1] Edmond Pilon, "Une grande amitié," *Revue Politique et Littéraire*, Nov. 19, 1932.

and extremely jealous, made a remark strikingly like that of the Pharisee in the Gospels. "Monsieur Fermat," he said, "is a Gascogne. I am not," meaning to say that the gentleman was a braggart. To this accusation the naive and affable scholar humbly replied, "Monsieur Descartes would not have esteemed me so little if I had not esteemed myself less."[2] Not Descartes with all his philosophical rigor, nor Pascal himself in a moral and religious crisis, attained such absolute detachment and complete separation from human honors with their noise and smoke.

Fermat's intimate friends were few. Only Blaise Pascal, Carcavi, and Father Mersenne gained the sanctum of this great soul. Pascal he loved dearly because of their similar views in many fields and because of the gentleness of his spirit, so like his own. Pierre Carcavi, librarian of Colbert and confrere at the Parliament, was the "dépositaire" of most of his beautiful writings. To Father Mersenne, however, he wrote most of his letters, delighting in the sympathetic and enthusiastic reception the "Reverend" Father gave them. Father Mersenne, more than anyone else in the century, realized the rare genius of this man and spread his name throughout the intellectual world.

When death came, it fittingly came quickly to such an active person. Fermat, on January 9, 1665, made a report of an important process to the Parliament. Three days later he was dead. In the *Journal des Savants* of February 9, 1665, there appeared the following eulogy of the great man.[3] "We have learned with much sorrow of the death of Monsieur Fermat, Councilor in the Parliament of Toulouse. His was one of the most beautiful spirits of the century, a genius so universal that if all the scholars had rendered testimony of his extraordinary merits one would have some difficulty believing all the things said of him, for nothing could cut short his eulogy."

Samuel had a very hard time collecting his father's works for publication, for they lay in every part of the mathematical world and were written in Spanish, German, Latin, Greek, English, and French. The great man had never kept a copy of his proofs for himself, for as he often said, "I content myself with having discovered the truth and knowing the means of proof if I should ever lose it again."[4] It is fortunate in one way that Fermat left the only record of his discoveries in his letters to the eminent mathematicians of the day, for in studying these manuscripts we do not investigate didactic documents, but the vibrant personality of a great genius which is apparent in the spontaneous words to his friends.

Today in the ancient city of Toulouse there rest in honored memory two of the greatest geniuses of all time, St. Thomas Aquinas and Pierre Fermat—one honored by the world of philosophy, the other by the world of science. There are many striking comparisons which could be drawn between the two great scholars, for example, the vast field covered by each in his own realm and the multiplicity of duties during life. But that which is most interesting, because of its rarity, is the deep humility possessed by each great mind. This quality, so unique in the character of man, was produced by a childlike simplicity and complete submission of their lives to God's will. Fortunately, Samuel Fermat was able to include in the works of his father a single Latin poem written by the latter to Jean Balzac, a famous author of the day, who seemed to be enamored of his own brilliance. With the permission of the gentlemen of mathematics we will conclude this life of Fermat with parts of his "Cede Deo"; our excuse for placing it in a mathematics journal is that it gives us a glimpse of Fermat that is unique, for in no other place do we find him revealing his heart and not his intellect. And we must know not only Fermat

[2] *Ibid.*
[3] Pierre Fermat, *Varia Opéra*, p. 14.

Pierre Fermat, *Oeuvres*, Vol. II.

the mathematician to appreciate his universal genius, but Fermat the poet, philosopher, and deeply religious gentleman.

Alas, 'tis a pity the sharpness of your intellect
Has taught you to prefer vain colors to the true light.
Why does your unhappy reason carry on its war,
And seek to destroy by the sharpness of its tongue, God?
Why have you bound your imagination with fear of false images?

. . . .

Yield to God, God whom all the heavens proclaim,
God before whom all nature is subservient,
Whom the earth and the sea and the fierce indomitable winds obey.[5]

. . . .

Fermat's method of infinite descent

by Howard Eves, University of Maine, Orono, Maine

Diophantus, the famous Greek number theorist of antiquity, wrote at least three mathematical works, of which his *Arithmetica* is by far the most important. The *Arithmetica* is an analytical treatment of algebraic number theory and marks the author as a genius in this field. Only six of the original thirteen books of the *Arithmetica* are extant, and these six books are devoted to the solution of about 130 number problems of considerable variety and degree of difficulty. There have been many commentaries of the *Arithmetica*, but it was Regiomontanus who, in 1463, called for a Latin translation of the extant Greek text. The call was met in 1575 by Xylander (the Greek name assumed by Wilhelm Holzmann, a professor of mathematics at the University of Heidelberg), who made a meritorious translation with commentary. This Xylander translation was later used by the Frenchman Bachet de Méziriac when, in 1621, he published the first edition of the Greek text along with a Latin translation and notes.

Pierre Fermat secured a copy of the Bachet edition of Diophantus' *Arithmetica* and used the copy as a combination textbook and notebook. Many of Fermat's contributions to the field of number theory occur as marginal statements inserted in his copy of the *Arithmetica*. In 1670, five years after Fermat's death, these marginal notes were incorporated in a new, but unfortunately carelessly printed, edition of the *Arithmetica*, brought out by Fermat's son, Clément-Samuel. In this edition we find, accompanying Problem 26 of Book VI, the following marginal note by Fermat. "The area of a right triangle whose sides are rational numbers cannot be a square number. I have obtained a proof of this theorem only after extensive and arduous effort. I here reproduce the proof, since the procedure used will make possible wonderful progress in number theory." Then follows an indication of the proof.

The procedure alluded to by Fermat is ingenious, and has since become known as *Fermat's method of infinite descent*. The method apparently was used with success by Fermat on a number of occasions. For example, in one of his letters to Roberval, Fermat describes the difficulties he experienced in trying to establish a celebrated conjecture made by Bachet that every positive integer can be written as the sum of at most four squares, and in the letter

[5] Pierre Fermat, *Oeuvres*, Vol. I, 396–98.

Fermat says that he finally succeeded by the use of his favorite method of infinite descent. Again, in 1897 a paper was found in the library at Leyden among the manuscripts of Christiaan Huygens, in which Fermat describes the method of infinite descent.

The method of infinite descent is particularly useful in establishing negative results. In outline the method is this. To prove that there do not exist positive integers $a, b, c, \cdots$ satisfying a relation $R(a, b, c, \cdots)$, assume the contrary. On this assumption show that $R(a_1, b_1, c_1, \cdots)$ holds, where $a_1 < a$. Then in like manner we may show that $R(a_2, b_2, c_2 \cdots)$ holds, where $a_2 < a_1$, and so on ad infinitum. But, since there are only a finite number of positive integers less than a, this is impossible. We are thus led to a contradiction, whence we conclude that the relation $R(a, b, c \cdots)$ is not satisfied by positive integers $a, b, c, \cdots$.

To clarify the Fermat method of infinite descent, let us consider a simple application of the method. Let us prove by the method that $\sqrt{2}$ is irrational. Suppose, on the contrary, that $\sqrt{2} = a/b$, where a and b are positive integers. Now

$$\sqrt{2}+1 = 1/(\sqrt{2}-1),$$

whence

$$\frac{a}{b}+1 = \frac{1}{\dfrac{a}{b}-1} = \frac{b}{a-b},$$

and

$$\sqrt{2} = \frac{a}{b} = \frac{b}{a-b} - 1 = \frac{2b-a}{a-b} = \frac{a_1}{b_1}, \text{ say.}$$

But, since $1 < \sqrt{2} < 2$, after replacing $\sqrt{2}$ by a/b and then multiplying through by b, we have $b < a < 2b$. Now, since $a < 2b$, it follows that $0 < 2b - a = a_1$. And since $b < a$, it follows that $a_1 = 2b - a < a$. Thus a_1 is a positive integer less than a. By a reapplication of our procedure we find $\sqrt{2} = a_2/b_2$, where a_2 is a positive integer less than a_1. The process may be repeated indefinitely. But the positive integers cannot be decreased in magnitude indefinitely. It therefore follows that our original assumption that $\sqrt{2} = a/b$, where a and b are positive integers, is untenable. That is, $\sqrt{2}$ is irrational.

"That the whole is greater than its parts is not an axiom, as that eminently bad reasoner, Euclid, made it to be . . . . Of finite collections it is true, of infinite collections false."—*C. S. Peirce.*

"A good numeration, accompanied by a good notation, is essential for proficient work in numbers. We are told that the Yancos on the Amazon could not get beyond the number three, because they could not express that idea by any phraseology more simple than Poetterrarorincoaroac."—*F. Cajori.*

Edited by Donovan A. Johnson, University of Minnesota High School, Minneapolis, Minnesota

Editor's note: New curricula in mathematics are suggesting more advanced topics to be included in high school mathematics courses. The following articles contain unique presentations of ideas related to complex numbers. You are urged to send the editor ideas that you have found successful when teaching mathematical concepts. These reports need not be lengthy or profound. Let us hear from you soon about your favorite "trick of the trade."

Rotation of axes with complex numbers

by Robert M. Rippey, Lyons Township High School, La Grange, Illinois

The derivation of the formula for the rotation of axes is simplified considerably when complex numbers are used for the new co-ordinates of the point. The proof below allows the student to make use of what he already knows about complex numbers. The proof also has the advantage of being independent of a cumbersome diagram. At the same time, it gives the student an opportunity to use mathematical ideas in a way that illustrates the depth and power of these ideas. The following proof has been used successfully with a class of gifted high school juniors which I have had the good fortune to teach.

1. If the vector $x+iy$ is drawn from the origin, then rotation of the axes through a positive angle θ is equivalent to rotating the vector through an angle $-\theta$.
2. The rotation of a vector through an angle θ is accomplished by multiplication by *Cis* θ where *Cis* θ is $(\cos\theta+i\sin\theta)$.
3. The new co-ordinates of the point (x_1, y_1) are found by multiplying the vector $(x+iy)$ by the vector $(\cos(-\theta)+i\sin(-\theta))$.

$$
\begin{aligned}
x_1+iy_1 &= (x+iy)(\cos(-\theta))+i\sin(-\theta)\\
&= (x+iy)(\cos\theta-i\sin\theta)\\
&= x\cos\theta-ix\sin\theta+iy\cos\theta\\
&\quad -i^2y\sin\theta\\
&= x\cos\theta+y\sin\theta\\
&\quad +i(y\cos\theta-x\sin\theta)\\
x_1 &= x\cos\theta+y\sin\theta\\
y_1 &= y\cos\theta-x\sin\theta
\end{aligned}
$$

Implicit throughout this proof is the fact that we have a one-to-one correspondence between each of the following: ordered pairs of real numbers, co-ordinates of points in a plane, vectors centered at the origin, and complex numbers.

It is this one-to-one correspondence which permits us to shift from the ordered pair (x, y) to the point (x, y) and from this point (x, y) to the vector $(x+iy)$. For further information about complex numbers the reader is referred to the *Appendices* of the Report of the Commission on Mathematics.

Complex numbers and vectors in high school mathematics

by Arthur H. Pedley, Canajoharie Central School, Canajoharie, New York

Complex numbers and their trigonometric forms have not received their fullest use in secondary schools. Too often they are introduced and perhaps extended to the point that shows students how complex numbers are plotted on the complex plane—then the subject is dropped. Some textbooks carry the matter further and show how complex numbers may be added and subtracted graphically and then couple this with an introduction to the trigonometric forms of complex numbers. Very few textbooks have been found that carry this graphic analysis far enough to show students how to apply this knowledge to the addition and subtraction of vectors and the resulting possibilities for new ways to solve familiar problems in mathematics as well as in physics.

To illustrate the use of complex numbers and their trigonometric forms in the solutions of two common problems in plane trigonometry, three areas in complex numbers and vectors should be clear.

First, the student should learn to plot complex numbers on the complex plane and represent a complex number by its trigonometric form $r(\cos A+i \sin A)$, where r is the modulus and A is the angle between the positive side of the real axis and the modulus. This means that point P (Fig. 1) can be represented by both the complex number $a+bi$ and by its trigonometric form $r(\cos A+i \sin A)$, where $r=\sqrt{a^2+b^2}$ and $A=\tan^{-1} b/a$ (where $\tan^{-1} b/a$ is used to represent arctangent b/a).

Second, the student should be familiar with the law of the parallelogram in vector analysis. The sum of two vectors whose direction, magnitude, and sense are known, can be found by forming the parallelogram whose adjacent sides are the two known vectors and whose diagonal drawn from the vertex formed by these two vectors is the sum vector. This means that in parallelogram $ABCD$ (Fig. 2) $\overrightarrow{AB}$ represents a vector with length a, $\overrightarrow{AD}$ represents a vector with length b, and $\overrightarrow{AC}$ represents the sum vector. Since the magnitude, direction, and sense of both vector $\overrightarrow{AB}$ and vector $\overrightarrow{AD}$ are known, orienting the parallelogram as illustrated with vertex A on the origin of the complex plane and side AB

Figure 1

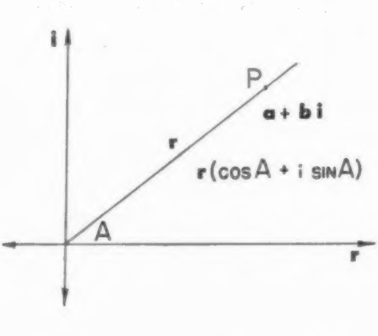
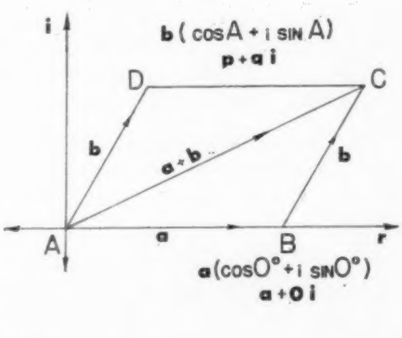


Figure 2



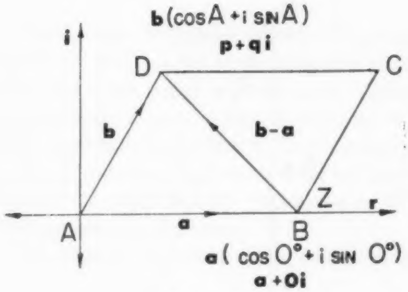


Figure 3

along the positive side of the real axis allows point B to be represented by $a(\cos 0°+i \sin 0°)$ and point D by $b(\cos A+i \sin A)$. Each of these trigonometric forms of complex numbers can be expressed in their complex number equivalents, point B by $a+0i$ and point D by, say, $p+qi$. Upon addition, the resulting complex number represents point C of diagonal AC. Hence point C is represented by the complex number $(p+a)+qi$ and the magnitude of vector $\overrightarrow{AC}=\sqrt{(p+a)^2+q^2}$ and its direction or angle $CAB=\tan^{-1}q/p+a$. With this information, triangle ABC can be completely solved.

The third and final area in complex numbers and vectors that should be clear is that the diagonal DB of parallelogram $ABCD$ (Fig. 3) is the difference vector of the two vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$. (Notice the correspondence between Figure 2 and Figure 3.) The direction, magnitude, and sense of $\overrightarrow{DB}$ as a vector can be taken from B to D or from D to B. If taken from B to D, vector $\overrightarrow{BD}$ represents vector $\overrightarrow{AD}$ minus vector $\overrightarrow{AB}$. If taken from D to B, vector $\overrightarrow{DB}$ represents vector $\overrightarrow{AB}$ minus vector $\overrightarrow{AD}$.

Relating Figure 2 and Figure 3 and taking the direction, magnitude, and sense of the difference vector $\overrightarrow{DB}$ of Figure 3 from B to D, so that it represents the vector $\overrightarrow{AD}$ minus $\overrightarrow{AB}$ with magnitude $\sqrt{(p-a)^2+q^2}$ and its direction angle $Z=\tan^{-1}q/p-a$. Now the supplement of angle Z is angle ABD. With this information triangle ABD can be solved.

Assuredly the difference vector $\overrightarrow{BD}$ could have been taken with the opposite sense so that it would represent the vector $\overrightarrow{AB}$ minus $\overrightarrow{AD}$. Solutions, however, appear simpler if they are taken as illustrated.

Two problems have been selected to illustrate how this information can be applied to solve problems met by every student of plane trigonometry.

The first problem was taken from a New York State Regents Examination in Trigonometry and reads: Two forces of 33 pounds and 47 pounds, respectively, act on a body at an angle of 45° 40′ with each other. Find, to the nearest ten minutes, the angle formed by the resultant and the greater force.

Since the common method of solution is based upon the law of tangents, this method will be compared with the solution using complex numbers and their trigonometric forms.

Solution by the law of tangents:

$$\frac{c+a}{c-a}=\frac{\tan \frac{1}{2}(C+A)}{\tan \frac{1}{2}(C-A)}$$

$$\frac{40}{7}=\frac{\tan 22° \, 50'}{\tan \frac{1}{2}(C-A)}$$

$$\tan \tfrac{1}{2}(C-A)=\frac{7 \tan 22° \, 50'}{40}$$

$$\log \tan \tfrac{1}{2}(C-A)=8.8673-10$$

$$\tfrac{1}{2}(C-A)=4° \, 13'$$

$$\text{angle } CAB=18° \, 37'$$

Correct answer: 18° 40′

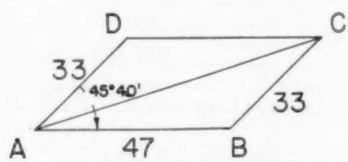


Figure 4

Solution by complex numbers and their trigonometric forms:

Point B:

$$47(\cos 0°+i \sin 0°)\equiv 47+0i$$

Point D:

$33(\cos 45° 40' + i \sin 45° 40')$
$$\equiv 23.0604 + 23.6049i$$

Point C: $70.0604 + 23.6049i$

$$\text{angle } CAB = \tan^{-1} \frac{23.6049}{70.0604} = 18° 37'$$

Correct answer: 18° 40′

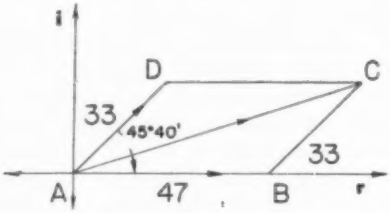


Figure 5

The second illustrative problem shows the use of complex numbers and their trigonometric forms as applied to the difference vector. This problem was taken from a New York State Regents Examination in 11th-Year Mathematics and reads: "In triangle ABC, $BC=84$, $AC=36$ and angle $C=93°$. Find angle A to the nearest minute."

Again, the common method of solution is based upon the law of tangents and hence this method will be compared with the solution using complex numbers and their trigonometric forms.

Solution by the law of tangents:

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}$$

$$\frac{5}{2} = \frac{\tan 43° 30'}{\tan \frac{1}{2}(A-B)}$$

$$\tan \tfrac{1}{2}(A-B) = \frac{2 \tan 43° 30'}{5}$$

$$\log \tan \tfrac{1}{2}(A-B) = 9.5792 - 10$$

$$\tfrac{1}{2}(A-B) = 20° 47'$$

$$\text{angle } A = 64° 17'$$

Correct answer: 64° 17′

Solution by complex numbers and their trigonometric forms: Notice the orienta-

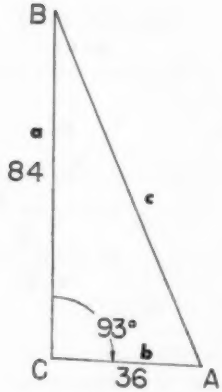


Figure 6

tion of the triangle ABC so that vector $\vec{c}$ equals vector $\vec{a}$ minus vector $\vec{b}$.

Vector $\vec{b}$: $36(\cos 0° + i \sin 0°) \equiv 36 + 0i$

Vector $\vec{a}$: $84(\cos 93° + i \sin 93°)$
$$\equiv -4.3932 + 83.8824i$$

Vector $\vec{c}$: $-40.3932 + 83.8824i$

Direction of vector $\vec{c}$ is indicated by angle P.

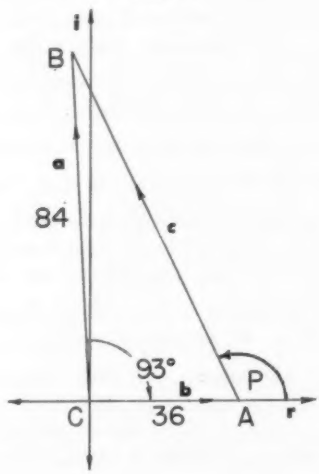
$$\angle P = \tan^{-1} \frac{83.8824}{-40.3932} = \tan^{-1} -2.0766$$

angle $P = 115° 43'$

angle $A = 64° 17'$

Correct answer: 64° 17′

Figure 7



These two problems merely illustrate what can be done by using complex numbers and their trigonometric forms. Other applications should certainly become evident, and students should be encouraged to be alert for applications to other problems in the mathematics classroom as well as in classes in physics.

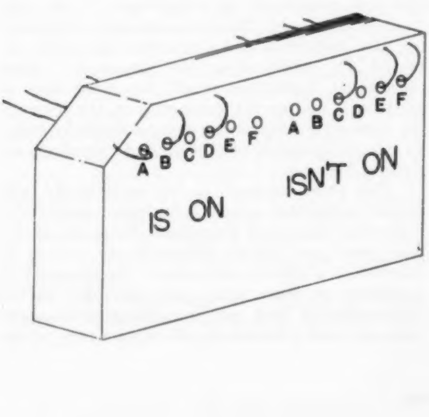
The school in which these methods of solution have been taught is a small one with a graduating class of less than one hundred. Nevertheless this approach has always aroused the interest of students of trigonometry. With the reported increase in the use of complex numbers in college physics courses, vector solutions may well be of considerable help to students who enter college.

Letter to the editor

Dear Editor:

I am a student at Base Line Junior High School, and my algebra teacher, Mr. Norton, told us about the article "A Punchcard 'Adding' Machine Your Pupils Can Build," by Larew M. Collister, in your October magazine. He had forgotten to bring it, but he remembered that there were cards with notches on the top edge corresponding to their binomial notations. Wires are put through the holes, and the card with the number someone has chosen is found. We worked on binomial and other systems last year, so we were familiar with the working of the charts.

That night I worked out a method that seemed to fill most of these requirements and later made a deck of these cards which worked quite well. I believe that this method is much easier. It involves only one shaking; you don't have to think out whether to take the ones that fell out or the ones that remained; and only one falls on the floor.



Here's how you would find 11. The person who has chosen the number tells you that it is on chart A, so you put a wire through hole A on the "Is on" side. It's on B too, so you put one in B on the same side. Since the number doesn't appear on chart C, put another wire in hole C on the "Isn't on" side. After you do this with the other charts, hold the wires, shake the deck, and a card marked "11" will fall out.

In writing the binomial notation on the cards, the left-hand side of the cards use a "V" for a one and a "0" for a zero and the right side has a "0" for one and "V" for zero. The digits are reversed on both sides so hole A will be at left and F at right in normal reading order.

The reason for cutting notches twice for each number is that, for instance with eleven, using only the left side, not only the eleven card would fall out, but also seven others (15, 27, 31, 43, 47, 59, 63). The wires in the "Isn't on C, E, and F" holes hold the other cards in the deck.

This device demonstrates sets, can be considered an "adding machine," and may be used to sort the cards in the same way as Mr. Collister's deck.

Sincerely yours,
HU McCULLOCH
Boulder, Colorado

Reviews and evaluations

Edited by Kenneth B. Henderson, University of Illinois, Urbana, Illinois

BOOKS

Elements of Calculus and Analytic Geometry, George B. Thomas, Jr. (Reading, Mass. Addison-Wesley Publishing Co., Inc., 1959). Cloth, x+580 pp., $7.50.

Elements of Calculus and Analytic Geometry is based on Thomas's earlier text, *Calculus and Analytic Geometry*, and contains material concerned with analytic geometry and with the calculus of functions of one independent variable. The language is particularly clear and the illustrations are well chosen. The format is excellent; the use of formulas in display boxes, italics, and varied type styles all enable the author to enforce his points of emphasis. Miscellaneous problems at the end of each chapter give exercises to challenge the better students.

An illustration of the care with which Professor Thomas approaches a topic is found in the first chapter where, in accordance with the best current usage, function is restricted to single-valuedness and is defined as a set of ordered pairs and the extent of domain and range is duly noted (page 17). The problem of multiple-valuedness arising from the consideration of the inverse function is not by this device wholly averted, however, as is seen in the solution of problems on page 24. An early introduction to the graphs of the absolute value function and the "greatest integer function" provides the student with graphs illustrating important properties. Professor Thomas brings in the $\epsilon-\delta$ concept of limit and presents an interesting and helpful analogy to a manufacturer offering a product within "tolerance limit" specifications. Continuity, with an early distinction between ordinary continuity and uniform continuity, is similarly treated with great care in the second chapter. The usual type of applications follow in the third chapter: related rates, curve tracing, maxima and minima. Rolle's Theorem and the Mean Value Theorems with applications conclude the chapter.

Analytic Geometry is studied in the next two chapters. The fundamental principle of duality between a *curve* and the equation determining it is well stated here. It appears that the term *graph* introduced earlier (page 18) should be related to *curve*. Perhaps it would have been preferable to define *graph* more precisely as "the set of all points *and only those points* whose coordinates form a pair which belong to the function," to make graph conform to the later notion of curve. The same remark applies to the definition of *graph of a polar equation* on page 227. Some standard trigonometric formulas are developed in an interesting fashion using the coordinate system. The conic sections are thoroughly discussed, and the equations for rotation are related to the trigonometric addition formulas. The invariants under rotation and the information yielded by them are developed here. Newton's method for approximating roots of equations is treated in an excellent fashion, and the remarks on the accuracy of the method and its limitations are quite appropriate. Polar co-ordinates are discussed in the following chapter.

Integration is introduced in the sixth chapter by first studying the indefinite integral. Differentiation and integration of the sine and cosine functions are treated together. After computing the area under a curve, the fundamental theorem of the calculus is studied. The trapezoidal rule for approximating the integral is quite naturally inserted at this stage. The usual applications to volumes and length and surface area follow. The cuts illustrating the volumes are particularly good. Applications to physics are given in the next chapter. Centroids, the theorems of Pappus, moments of inertia, the radius of gyration, hydrostatic force, and work are treated. Remarks such as that relating moments to statistics should prove interesting and helpful.

Transcendental functions are differentiated and integrated in Chapter Eight. The analysis of the inverse trigonometric functions is excellent, and the rationale of the choice of range of principal values is particularly well developed. In the next chapter, techniques of integration are given due consideration. In addition, improper integrals and Simpson's rule for approximating a definite integral appear here. The properties of vectors and parametric representation are combined to advantage in the last chapter. Professor Thomas's examples illustrating the fact that the Cartesian equation obtained by eliminating the parameter from parametric equations might be comprised of additional points not belonging to the locus of the parametric equations are particularly good. Vector components of motion and curvature are treated here also.

This book appears to be particularly well suited to the high school calculus course offered under the Advanced Placement Program. If this text were used in an introductory course to calculus in a college curriculum, the material on functions of more than one variable—partial differentiation and multiple integrals—usually given in such a course would of necessity be de-

ferred to a subsequent course.—*Harold Tinnappel, Bowling Green State University, Bowling Green, Ohio.*

An Introduction to Mathematics, Donald W. Western and Vincent H. Haag (New York: Henry Holt and Company, Inc., 1959). Cloth, xi+580 pp., $7.50.

Our net impression of this book as a text for college freshmen is a favorable one. We think it a worthwhile addition to the growing number of texts which deviate radically from the traditional college algebra, trigonometry, and analytic geometry pattern. The essentials of each of these fields are, however, included in the text even though the number of pages devoted to the aforementioned topics is certainly less than it has been in the traditional sequence.

The book is written on much the same pattern, but on a somewhat less abstract level, than the Allendorfer and Oakley text with which the reader may be familiar (reviewed in the December 1955 issue of THE MATHEMATICS TEACHER). The research mathematician will probably not be entirely in sympathy with the oftentimes heuristic argument, wordy exposition, and the many examples which appear to impede logical development. But as a text for the average student, it is the better, we feel, for proceeding at a deliberate, if not always rigorous, pace.

To give the reader an idea of the content of the book, we list the chapter headings. 1. Logic, 2. The Natural Numbers, 3. The Complex Number System, 4. Extensions of Logic to Algebra, 5. Numerical Calculations, 6. Algebra of Sets, 7. Analytic Geometry and Functions, 8. The Probability Function, 9. Circular Functions, 10. Lines and Vectors, 11. Conic Sections, 12. Limits of Sequences and the Definite Integral, 13. Limits of Functions and the Derivative, 14. The Calculus, 15. Curve Tracing, 16. Matrices, 17. Analytic Geometry of Three Dimensions. Since this is more material than can be covered in a year, the authors suggest in the preface that a terminal course consist of chapters 1–14 and that a prescience course consist of Chapters 1–11 and 15–17.

We are pleased to see the new topics for this level found in the book, but we are not completely satisfied with the organization of the material, particularly from Chapter Seven on. There is a lack of continuity in this part of the text in our opinion. We prefer to see the analytic of two and three dimensions treated simultaneously and the direction cosines and direction numbers used in two dimensions. For the prescience student we would prefer that some of the space devoted to calculus be spent on more space geometry. We feel this would be of more value than precise definitions of continuity and the Fundamental Theorem of the Calculus are going to be to the student taking the suggested terminal course.

We are pleased with the chapter on trigonometry. One reason is that it contains only four pages devoted to the traditional triangle problems. The treatment accorded graphing trigonometric functions also appeals to us.

The inclusion of matrices and their application to solving systems of equations is long overdue in books at this level. In the analytic geometry material on conics, we were pleased to find a brief discussion of Dandelin spheres.

We realize it is currently fashionable to emphasize the function concept. However, we wonder whether making the notion appear more sophisticated than it is—at least in the reviewer's opinion—by introducing sometimes useless distinctions between equations, relations, and functions and introducing new terminology such as "*f*-function" actually results in greater clarification of the concept.

There are errors here and there, some probably due to poor proofreading. On page 21, Example 5 should read "Definition 7(c)" instead of "7(d)." On page 318, we have proven, in effect, that the projection of $\overrightarrow{A}$ on $\overrightarrow{B}$ equals the projection of $\overrightarrow{B}$ on $\overrightarrow{A}$ for arbitrary vectors. We also take exception to the mathematical induction proof, page 53, of "$n(n+1)$ is a multiple of 2." The inductive assumption reads, "Assume $P(k)$ true: $k(k+1)$ is a multiple of 2; that is, $k(k+1)=6c$ for some natural number c." The 6 is not a misprint, for the final line of the proof is, "$(k+1)(k+2)=2[3c+(k+1)]$." Unfortunately there are infinitely many values of k for which there exists no natural number c satisfying the inductive assumption and the final equation in the proof. This example will hardly serve as model proof by induction.

For the most part, the authors are successful in making use of the principles of logic of Chapter One in later chapters. A liberal number of exercises requiring proofs are scattered through most of the sets of problems. Another desirable feature of the text is the inclusion of graphs in the answer section. But where proofs are required, these are not given in the answer section. Consequently the book can be utilized to its fullest only in the hands of a competent instructor.

The text contains interesting, pertinent historical material, including mention of the proof of the impossibility of solving the general polynomial equation of degree five and higher by formula. Also, many sets of problems contain exercises requiring reading in other books and journals.—*Arnold Wendt, Western Illinois University, Macomb, Illinois.*

Introductory Calculus, Donald S. Richmond (Reading, Mass.: Addison-Wesley Publishing Co., Inc., 1959). Cloth, xv+205 pp., $5.50.

Professor Richmond has long enjoyed a reputation for singularly effective exposition of topics in introductory mathematics. The present volume is a good example of why this is so. It succeeds admirably in achieving its objective of giving a clear, modern presentation of

calculus for students in grade twelve or for college freshmen. The emphasis is on basic concepts, such as *continuity, limit, function*, etc., and these ideas emerge with clarity.

This is a short introduction to calculus; it does not cover an entire freshman-sophomore college course (as, for example, G. B. Thomas' *Calculus and Analytic Geometry* manages to do); in particular, there are no chapters on convergence of infinite series, or the algebra of vectors, or several independent variables. What is included has been carefully chosen to give the beginner a correct idea of the nature and uses of calculus.

This is a gentle introduction, a pleasant one, an accurate one, and a "modern" one. With luck, it may help to dispel the forbidding aura with which the study of calculus has been unjustly cloaked.—*Robert B. Davis, Syracuse University, Syracuse, N.Y., and Yale School Mathematics Study Group, New Haven, Conn.*

Learning to Use Arithmetic, F. Lynwood Wren, *et al.* (Boston: D. C. Heath and Company, 1959). Cloth, vi+378 pp., $2.68.

Although this book follows a traditional outline of the content for grade seven, the approach to the subject matter is more modern than that of many of the seventh-grade textbooks on today's market. New topics are approached to develop meaning first and then definitions and skills. The authors have done a fine job, in general, of taking the mystery out of mathematics and replacing the mystery with meaning. For instance, there are discussions of the meanings of addition, subtraction, multiplication, and division, followed by pages on "When Do You Use Addition?" ". . . Subtraction?" and so on, with appropriate illustrative exercises.

Alternate methods for solution of various types of problems are provided so that students may see that there are different ways for solving problems. They may then choose the method they consider best. Considerable emphasis also is placed upon estimation of answers and the use of the estimates to judge the reasonableness of solutions to problems. Sets of exercises and questions are spaced throughout the text to help students learn the important skill of estimating answers.

The meaningful approach is illustrated further by the care that the writers of this text have taken to eliminate some of the "dirty words" (mathematically speaking) ordinarily found in seventh-grade textbooks. For example, the words "carry," "borrow," "cancellation," and "reduce" appear *only once* as the ideas traditionally associated with each of these terms are explained as meaningful mathematical processes. The "dirty words" are then dropped in favor of more meaningful expressions. On the other hand this same precision in language is violated by some of the definitions found in the glossary of the book. For example, the *base* of a geometric figure is defined as "the side on which the figure rests" (p. 369). An *angle* is "the figure formed by two intersecting lines" (p. 369). At least *four* angles are formed by two intersecting lines. The authors also cling to the word "invert" in preference to taking the "reciprocal" of the divisor in dividing fractions. To *invert*, say the authors, is "to turn upside down" (p. 370). The definitions above could have been more percisely stated.

In the chapter on geometry, the authors vary from more standard practice by developing the formulas for the areas of the common rectilinear figures and the circle in this order: rectangle, right triangle, square, circle, parallelogram, and the general triangle. This sequence appears to be no better than, and perhaps not as good as, the more traditional sequences. There are better ways, in the opinion of this reviewer, to develop the area of the circle than that shown in this book (pp. 320–21). The discussion of *pi* also leaves much to be desired. The implication seems to be that *pi* is the rational number 3.14 rather than an irrational number (pp. 309, 310, 321). Volumes are not considered at all in this text.

Although there is reasonably good coverage on each topic in the book, a few opportunities were missed for tying the topics more closely together. For instance, in explaining the multiplication of fractions and whole numbers the commutative principle of multiplication is used, but this important principle is not generalized for all numbers known to the student (p. 80). In another case, tests of divisibility by 2, 3, 4, 5, 6, 8, and 9 are given (p. 43), but this information is not related to its use in changing fractions to lowest terms or to finding the least common denominator of a group of fractions.

Special features of the text beyond those mentioned above include a glossary, pages devoted to the history of mathematics, mathematical recreations, and a wide variety of types of tests. There appears to be an over-categorization of evaluative instruments in the book which has pages devoted to "Practice to Remember," "Review," "Maintenance Test," "Diagnostic Test," "Readiness Tests," "Test on Problems," "Test on Meanings," and "Test on Skills."

Summarizing, it should be said that the principal strength of this book resides in the care which the authors took to develop meanings and to take the mystery out of mathematics. Although there are weak points in the text, its strengths outweigh its weaknesses. The book is certainly worthy of consideration for "the middle seventy per cent" of seventh grade youngsters. It would probably have to be supplemented considerably for a superior class in seventh grade mathematics.—*Eugene P. Smith, Chairman of Secondary Mathematics, Wilmington Public Schools, Wilmington, Delaware.*

Mathematics for the Academically Talented Student, edited by Julius H. Hlavaty (Washington, D. C.: National Education Association and National Council of Teachers of Mathematics, 1959). Paper, 48 pp., 60¢.

This report of a conference sponsored jointly

by the NEA project on the academically talented student and the NCTM is interesting, informative and encouraging. The booklet contains very little that is "new," but it is helpful to find assembled many concrete aids to thought and action. Administrators and mathematics teachers in the junior and senior high schools will find it worth reading.

The educators follow a safe and sane middle path, making no specific recommendations. However, they have prepared practical suggestions for placing the mathematically talented student, planning courses of study suitable for such students and providing them with a truly enriched extracurricular program. These suggestions are broad enough to assist administrators and teachers in either large or small schools.

Institutes, with which everyone is familiar, always prove beneficial, but much can be done on one's own time to "keep up" or "catch up" by worth-while reading. The many teachers who hesitate to accept the challenge presented by the talented student will be encouraged by the appendices which include an extensive subject-matter bibliography for teachers (with books you *haven't* read!); a list of paperback books; and something you have been looking for—a brief bibliography of reference and enrichment material graded to indicate interest and degree of reading difficulty.

Mathematics for the Academically Talented Student should be on *your* bookshelf. No better buy for sixty cents!—*Sister Marie Celeste, O.P., Trinity High School, River Forest, Illinois.*

Mathematics in Secondary Modern Schools, A Report Prepared for the Mathematical Association for Consideration by All Concerned in the Teaching of Mathematics in the Secondary Schools (London, England: G. Bell and Sons Ltd., 1959). Cloth, vi+221 pp.

This bulletin is a report by a subcommittee of the Teaching Committee of the Mathematical Association. The study began in 1946 with an interim report in 1949. This report is a companion report to the Primary Report and should be considered as a continuation of it. The report deals with mathematics instruction for ages 11 to 15 though not for the ablest of this age group.

One of the principles emphasized is the importance of fresh approaches at each level, of the need for activities in the pupil's sphere of experience and the treatment of mathematics as a functional subject at the present time, doing such things as weighing, measuring, and counting money. Chapter 3 deals with the practical topic as a teaching method. The authors emphasize that all pupils need actual experience and experimentation. The study of mathematics should be adjusted to fit the environment. Many potential topics are listed as illustrations, which should help teachers a great deal in instruction. The mathematics taught should also be adjusted to the major area of study—vocational, handicraft, nursing, etc.

Chapter 4 defines "remedial" as work needed for students who have missed instruction or who are far behind, while "revision" is that recently broached but not yet mastered. Attitude is very important. Where remedial work is needed because a student was not sufficiently experienced when it was first introduced, his added experience may require a presentation completely different from the first. Initiative must be left to the student; do not show him everything. In Chapter 5 it is recommended that homogeneous groups be established when possible and that emphasis in instruction be placed on the manner of doing as opposed to the result. It might also be possible to provide for student differences through time allowances and a variety of textbooks. Extended use gives better understanding than periods of isolated practice.

Part II outlines rather carefully the content to be taught and studied in the years 11 to 15. It is pleasant to note the emphasis on mathematics as a language, the idea of number systems, quantities that can be presented by numbers, and operations with all kinds of numbers. Other topics included are algebra, graphing, co-ordinate geometry, Euclidean geometry, volume, symmetry, loci, and trigonometry. Trigonometry is used to unify the algebra and the geometry. There is suggested work in surveying, navigation, mechanics, statistics, and perspective drawing.

It seems to the reviewer that this is a very helpful bulletin and that it emphasizes again and again the need for introducing things through a new approach, motivating the pupil, and developing attitudes conducive to good mathematics study by all.—*Philip Peak, Indiana University, Bloomington, Indiana.*

Modern Mathematics—Topics and Problems, Daymond J. Aiken and Charles A. Beseman (New York: McGraw-Hill Book Company, 1959). Paper, iv+140 pp., $1.48.

This is a workbook that, in my opinion, will be useful for presenting the modern point of view to any secondary or junior high class that either has studied elementary algebra from a traditional point of view or is currently studying elementary algebra from a traditional textbook. I believe that such classes will greatly increase their understanding of algebra by carefully working the exercises of the first six chapters under the direction of a skillful teacher who knows how to use modern ideas in a traditional setting.

The explanations of such concepts as set, universe, variable, solution set, ordered pair, function, intersection and union of sets are short, accurate, and for the most part understandable. The many varied exercises following brief explanations of each concept are designed to enhance understanding and lead to discovery of deeper insights. A few of the exercises involve knowledge and concepts that many ninth graders do not possess. The explanations of concepts and ideas as given in the workbook will require much amplification by the teacher. A teacher who knows the ordered-pair definition of a function will understand the following

(italics are mine): "If a relation contained two different ordered pairs with the same first *member*, then the vertical at that first *co-ordinate* would cut the graph of the relation in two distinct points." I doubt that students will understand this senteuce. It would be more understandable if the word "co-ordinate" were replaced with the word "member." However, the examples for study which follow this explanation make the idea clear, and by the time the exercises are completed I am sure the students will understand the idea even though they may not be able to state it accurately.

At the end of each chapter there is a page titled "Show That You Know." With this device the authors effectively summarize the main ideas of the chapter and at the same time provide the teacher with a convenient means of evaluating student progress. There are also tests at the end of each chapter.

The pages of this workbook are perforated for easy removal. There is a teacher's answer book which looks exactly like the student's book except that the answers are written in red in the spaces that are left blank for the student. The book contains a bibliography and an index.

There are ten chapters in the book. The titles of the chapters are. "Introduction to Sets"; "Universe and Variable"; "Conditions and Solution Sets"; "The Number Line"; "Ordered Pairs and Conditions with Two Variables"; "Functions"; "Intersection of Sets"; "Union of Sets"; "More About Sets and Set Operations"; and "First Notions in Statistics."

The teacher using this book needs to have a thorough understanding of set theory and needs to know how to use the ideas gained from this book to increase understanding and comprehension of usual work in secondary mathematics classes. Ideas gained from this book are of little use to secondary mathematics in and of themselves but can be a tremendous asset throughout the student's further study of mathematics if his teachers make proper, continuous, and intelligent use of them. Therefore, I highly recommend this book for use in mathematics classes of those teachers who are well trained in modern mathematics and who can use the ideas of modern mathematics to explain traditional subject matter.—*Clayton Gardner, White Plains High School, White Plains, New York.*

BOOKLET

Mathematical Needs of Prospective Students, College of Engineering, University of Illinois, Revised Edition. University of Illinois Bulletin, Vol. 56, No. 51, 1959. Distributed by the Office of Field Services, College of Education, University of Illinois, Urbana, Ill. Paper, 28 pp., 50¢.

The primary purpose of this bulletin is to give a digest of the minimum content of high school mathematics programs necessary for professional education in engineering. This is accomplished by listing topics in algebra, geometry, and trigonometry. The bulletin is directed to teachers who prepare students to enter engineering at the University of Illinois, but the list of mathematical competencies given in this bulletin will be *very* helpful to high school counselors and teachers of mathematics throughout the nation, regardless of the colleges their students plan to attend.

A unique feature of the bulletin is the fact that the selected topics are classified in two categories: (1) those which are considered indispensable and (2) those which are considered desirable but supplementary. This is excellent.

The bulletin emanates from the work of faculty members from the College of Engineering, the Department of Mathematics in the College of Liberal Arts, and the College of Education. The co-operative effort of various college staffs is uncommon, but certainly laudable.

This bulletin is a revision of a similar one published in 1951. Three changes are evident in the revision: 1. The addition of topics; 2. The shift of topics no longer considered indispensable to the supplementary list; 3. The alteration of the statement of certain topics or their placement in the list to indicate more clearly the nature of the knowledge signified.

The authors are to be commended for making these changes. They are cognizant that mathematics is a dynamic subject, characterized by growth and new applications, and that these may not be found in the curriculum. They also realize that the traditional curriculum fails to reflect adequately the spirit of contemporary mathematics.

The list of topics in secondary mathematics, an understanding of which is considered to be indispensable for prospective students in the College of Engineering, is the "meat" of the bulletin. A reiteration—this list should prove exceedingly helpful to all teachers of high school mathematics.

Under "Indispensable Topics" on page 9 one finds these four categories: *Basic Concepts, Algebra, Geometry, Trigonometry.* Twelve topics are listed under *Basic Concepts*. Several of these topics listed are skills and not concepts. This is just a matter of semantics. A better word could have been chosen for *Concepts*, possibly the word *Competencies.*

Other subjects briefly treated are "Programs for Gifted Students" and "Experimental Curricula." A more detailed discussion of these subjects would have been very advantageous.

One will find a list of books suggested for the high school library as sources of supplementary material. Many such lists can now be found, but the selection is excellent.

Sample questions from the placement examinations given entering engineering students at the University of Illinois are provided in the appendix to illustrate the general level and character of knowledge expected. These questions will be very welcome.

Paraphrasing, one might say, "Valuable information contained in a small package."—*Milton W. Beckman, U.S. Office of Education, Washington, D.C.*

Mathematics and industry—a report

by Marie S. Wilcox, Chairman, Committee on Co-operation with Industry

The Committee on Co-operation with Industry of the National Council continues to seek to publicize instances of co-operation between industry and students and teachers of mathematics on the local level.

The Industry Aids to Education Program opened in 1956 when the New England Council's Industrial Research Committee began to consider ways to strengthen New England's economic growth. The program has a full-time manager and with a grant from the Ford Foundation in 1958 has achieved many specific results. Those of interest to teachers and students of mathematics include:

The Lexington Plan, which is a co-operative industry education approach to attract new graduates into teaching. Arthur D. Little, Inc., Sanders Associates, and RCA are the participating companies.

A lecture series in Natick, Massachusetts, to be given by outstanding specialists from industry for teachers of science and mathematics.

Arrangements for distribution of surplus or used electronic parts to a number of schools far from industrial centers.

A study of educational film sponsorship by industry.

Mr. Austin W. Fisher, Jr., chairman of the program's steering committee, states that "... schools have needs which cannot be filled by money alone. These needs involve the knowledge of people and materials and information for teaching which industry can supply at relatively little cost from its already established productive resources."

Two New York City high schools have received excellent co-operation from IBM in connection with courses in Programming for an Automatic Digital Computer for superior twelfth-grade students. The schools are Bronx High School of Science and the William Howard Taft High School. The Watson Laboratory of IBM provided instructors to teach the high school teachers, an unlimited amount of materials of instruction, including all necessary manuals concerning the different machines needed in programming, IBM cards by the thousands for the students to punch their instructions and data, and as much machine time on the 650's as was needed for the students actually to discover whether their programs worked.

Teachers in these New York schools have been experimenting with courses to see how they can use the programming as a core around which to teach selected topics in mathematics. For example, in the theory of numbers, they now have system-

atic methods of locating prime numbers, the sum of the divisors of numbers, perfect numbers, etc. They can introduce iteration methods for the solution of algebraic equations, finding the nth roots of numbers, and can use infinite series to make tables of trigonometric functions.

In the West, the Kennecott Copper Corporation made a grant for a research study of the teaching of integrated mathematics in Roosevelt Junior High School and East High School in Salt Lake City, Utah. The company has shown a considerable interest in education and has a large research laboratory on the campus of the University of Utah.

In Wichita, Kansas, various industries supply speakers and act as hosts at the annual dinners for teachers of science and mathematics sponsored by the Wichita Professional Engineering Society. This society also assists by supplying speakers for various teachers' meetings.

Instances of co-operation continue to be prominent in Southern California. Numerous industries supply prizes and pay for dinners for winners in the annual science, engineering, and mathematics fairs. In local fairs, engineers from industry assist in the judging. The Jet Propulsion Laboratory provided speakers for seventeen high schools and junior college classes in mathematics and science in Pasadena. A special science seminar for capable students was given last year at Pasadena High School from 7:00 A.M. to 8:00 A.M., with a large number of speakers from science and industry. A large block of time was devoted to modern aspects of mathematics and an introduction to calculus. The Occidental Life Insurance Company and the Actuarial Society assist with judging and awards for the Occidental College Mathematics Field Day, a competition that arouses keen interest among high school mathematics students.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D.C.

NCTM convention dates

THIRTY-EIGHTH ANNUAL MEETING

April 20–23, 1960
Statler-Hilton Hotel, Buffalo, New York
Louis F. Scholl, Board of Education, Buffalo 2, New York

JOINT MEETING WITH NEA

June 29, 1960
Los Angeles, California
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D.C.

TWENTIETH SUMMER MEETING

August 21–24, 1960
University of Utah, Salt Lake City, Utah
Eva A. Crangle, Board of Education, Salt Lake City 11, Utah

NINETEENTH CHRISTMAS MEETING

December 27–30, 1960
Arizona State College, Tempe, Arizona
Lehi Smith, Arizona State College, Tempe, Arizona

Other professional dates

Mathematics Section, Maryland State Teachers Association

March 26, 1960
Towson State Teachers College, Towson 4, Maryland
W. Edwin Freeny, President, 507 Milford Mill Road, Pikesville 8, Maryland

The Association of Mathematics Teachers of New Jersey

March 26, 1960
Trenton State College, Trenton, New Jersey
Gail B. Koplin, 14 Schuyler Drive, Clark, New Jersey

Association of Teachers of Mathematics of New York City

March 26, 1960
Room 846, Washington Irving High School, 40 Irving Place, New York, New York
Lee Batch, New Utrecht High School, Seventeenth Avenue and Eightieth Street, Brooklyn, New York

April 30, 1960
Teachers College, Columbia University
Leona Freeman, Morris High School, New York, New York

Illinois Council of Teachers of Mathematics

March 26, 1960, East St. Louis, Illinois
April 2, 1960, Normal, Illinois
April 9, 1960, Macomb, Illinois
April 11, 1960, Charleston, Illinois
April 16, 1960, Carbondale, Illinois
April 30, 1960, Cicero, Illinois
T. E. Rine, Illinois State Normal University, Normal, Illinois

Mathematics Section, New York Society for Experimental Study of Education

April 8, 1960
Grace Dodge Hall, Teachers College, Columbia University, New York, New York
John A. Schumaker, Secretary, Montclair State College, Upper Montclair, New Jersey

The Ontario Association of Teachers of Mathematics and Physics

April 19–20, 1960
Ontario College of Education, Toronto, Ontario
Father John C. Egsgard, St. Michael's College School, Toronto 10, Ontario

Association of Teachers of Mathematics of New York State

April 21–22, 1960
Statler-Hilton Hotel, Buffalo, New York
Pauline Morris, 207-D Kenville Road, Buffalo 50, New York

The Nebraska Section of the National Council of Teachers of Mathematics

April 30, 1960
Lincoln Public Schools Administration Building, 720 South Twenty-Second Street, Lincoln, Nebraska
Monte S. Norton, 720 South Twenty-second Street, Lincoln, Nebraska

Association of Mathematics Teachers of Philadelphia and Vicinity

May 5, 1960
McCallister's, 1811 Spring Garden Street, Philadelphia, Pennsylvania
Alexander Beck, 1032 East Phil-Ellena, Philadelphia 50, Pennsylvania

Women's Mathematics Club of Chicago and Vicinity

May 7, 1960
Stouffer's Restaurant, Randolph and Michigan, Chicago, Illinois
Sarane Starr, 5424 South Cornell Avenue, Chicago 15, Illinois

Chicago Elementary Teachers' Mathematics Club

May 9, 1960
Private Dining Room, Toffenetti's Restaurant, 65 West Monroe Street, Chicago, Illinois
Ramona H. Goldblatt, Burley School, Chicago, Illinois

Twelfth Annual Institute for Teachers of Mathematics, sponsored by ATMNE

August 24–31, 1960
University of New Hampshire, Durham, New Hampshire
George R. Smith, Saint Paul's School, Concord, New Hampshire

An Important Question
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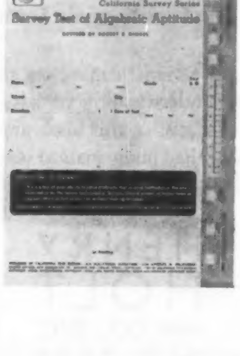
The new *Survey Test of Algebraic Aptitude*, devised by Robert E. Dinkel, is designed to help answer this question. It is nationally normed and specially devised for use in making important decisions for course selection and grouping purposes in the eighth and ninth grades. This test furnishes a single score from 40 minutes of testing time and may be either hand or machine scored. Part of the new *California Survey Series*, it is being published at this time in response to demands of educators all over the United States.



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